

Theory of pressure–strain-rate correlation for Reynolds-stress turbulence closures. Part 1. Off-diagonal element

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A theoretical calculation is made of (an off-diagonal element of) the pressure–strain-rate term $\rho_0^{-1}\langle p[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \rangle$ for a simple turbulent shear flow at high Reynolds number. This calculation is described as follows. (1) An expression for the pressure–strain-rate term is analytically derived in terms of measurable quantities (velocity spectra) – this derivation makes use of a cumulant discard. (2) It is proved that, to the lowest order in the spectral anisotropy, the (nonlinear part of) the pressure–strain-rate term is linearly proportional to the Reynolds stress. (3) A formula is derived for the constant of this proportionality (the Rotta constant) in terms of arbitrary velocity spectra. (4) This formula is used to analytically calculate Rotta’s constant, C_{zz} , for a class of models of velocity spectra (the variation of Rotta’s constant caused by variations in the spectral shapes is examined). (5) It is found that C_{zz} is surprisingly insensitive to the large-wavelength part of the spectrum. This insensitivity suggests that C_{zz} should not vary much from one turbulence application to another provided that the Reynolds number is very large. However, it is also shown that C_{zz} is unexpectedly sensitive to the short-wavelength part of the spectrum, and varies with Reynolds number when the latter is less than about 30.

The calculation is based on a straightforward solution of the Navier–Stokes equation to obtain formal expressions for \mathbf{u} and p . These expressions are then used to write the pressure–strain-rate in terms of a two-time fourth-order velocity correlation. The latter correlation is evaluated by a standard cumulant discard. Simplifying assumptions of the calculation are that average quantities vary little in space and time, and that the mean flow are unidirectional. These simplifications are made in order to emphasize the method of calculation and its details.

1. Introduction

An important term in the transport equation of Reynolds stress is the pressure–velocity correlation (the pressure–strain-rate term). Rotta (1951) proposed a plausible model for this term which has since been widely used in phenomenological turbulence closures (e.g. Reynolds 1976; Hanjalic & Launder 1972; Lumley & Khajeh-Nouri 1974; Launder, Reece & Rodi 1975; and many others). These closures have met with relative success (Reynolds 1976). However, due to uncertainties in various model terms of the closures as well as in experiments, it is difficult to predict just how accurate the Rotta model will be for a given turbulence application.

The purpose of this paper is to present an analytical calculation of the pressure–

strain-rate term directly from the Navier–Stokes equation. There are four interrelated goals of the calculations: (1) to derive analytically an expression for the pressure-strain term in terms of measurable quantities (covariant velocity spectra); (2) to prove that (to lowest order in the spectral anisotropy) the pressure–strain is linearly proportional to the Reynolds stress; (3) to derive a formula for the constant of this proportionality (the Rotta constant) in terms of arbitrary velocity spectra; and (4) to use this formula to calculate (estimate) analytically Rotta’s constant from straightforward assumptions about the general behaviour of velocity spectra in nearly homogeneous shear flows. With this calculation, we are able to examine the changes of Rotta’s constant caused by changes in the spectrum.

For the sake of simplicity, we will only consider an off-diagonal element of the pressure strain-rate tensor. The other elements can be readily calculated by the same method. We also, for simplicity, restrict ourselves to a uni-directional mean flow that varies only slowly in time on a Lagrangian integral time scale, and whose gradient varies only slowly in space.

1.1. *Plan of the calculation*

The plan of our calculation is to derive nonlinear expressions for the velocity fluctuations \mathbf{u} and pressure fluctuations p by a straightforward formal solution of the Navier–Stokes equation. The expression for p is not new and has been widely used (e.g. Chou 1945). The expression for \mathbf{u} is new in the present context. The derived expressions for \mathbf{u} and p allow us to relate the pressure–strain-rate tensor $\rho_0^{-1}(\langle p\nabla\mathbf{u}\rangle + \langle p(\nabla\mathbf{u})^T\rangle)$ (ρ_0 is the density and the superscript T denotes the transpose) to a two-point fourth-order velocity correlation. This correlation is then analytically evaluated in terms of single-point velocity covariances (i.e. the Reynolds stress). This evaluation is based on easily applied approximation methods that have been widely used in turbulence theory. The theme of our work, which is similar to the themes of Leslie (1973) and Herring (1974), is to apply turbulence theories to derive or improve phenomenological closure equations. However, our derivation is much less ambitious than the direct-interaction approximation (e.g. Leslie 1970; Herring 1974; Schumann & Herring 1976), and a knowledge of sophisticated turbulence theory is not needed. Because of its less ambitious nature, our calculation is simple enough to be entirely analytic. For example, our goal is not to calculate the energy spectra but, rather, to derive the pressure–strain-rate in terms of the spectra. In this derivation there occurs a constant C_{xz} (Rotta’s constant) which is explicitly expressed in terms of the energy spectra. This derivation permits an analytical determination of that constant.

The organization of the paper is as follows: In §2, an off-diagonal element of the pressure–strain-rate correlation, A_{xz}^N , is written in terms of the fluctuating velocity \mathbf{u} . A nonlinear expression for \mathbf{u} is derived from Navier–Stokes equation in §3. This \mathbf{u} expression allows us to write A_{xz}^N as a two-time fourth-order velocity correlation. In §4, A_{xz}^N is derived in terms of velocity spectra to general order in anisotropy. In §5, A_{xz}^N is explicitly related to the Reynolds stress to first order in the spectral anisotropy. This linear relationship resembles that of Rotta (1951). The proportionality constant (Rotta’s constant) is calculated in §6. The errors in that calculation are examined in §7. Also examined is the variation of C_{xz} with variations of spectra. A summary is given in §8.

1.2. Assumptions

To help clarify the theory, expose the underlying approximations, and, to allow more details to be included, we make the following simplifying assumptions: (a) the mean flow U has the idealized unidirectional form

$$\mathbf{U} = [U_0(z), 0, 0], \quad (1)$$

with Cartesian co-ordinate $[x, y, z]$; (b) $\partial\mathbf{U}/\partial z$ and all ensemble-average quantities (correlation functions) are assumed to vary only a little in space and time on scales L_0 and τ_L , respectively, where L_0 is the characteristic length scale of the energy-containing part of the spectrum and τ_L is the Lagrangian integral time scale; (c) we only calculate one off-diagonal element of the pressure-strain-rate tensor; and (d) the details of the calculation are worked out for high Reynolds number. The calculation can be readily generalized to more complex mean flow geometries, and to include all the elements of the pressure-strain tensor, but this is not done here. For low Reynolds numbers the expression for the pressure-strain-rate is only given in terms of the spectra.

2. Pressure-strain $\langle p\nabla\mathbf{u} \rangle$

The pressure-velocity correlation (pressure-strain-rate term) that appears in the stress transport equation is

$$\rho_0^{-1}\langle p[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \rangle,$$

where $\mathbf{u} \equiv \mathbf{u}(\mathbf{r}, t)$ is the fluctuating part of the fluid velocity at position \mathbf{r} at time t , $p \equiv p(\mathbf{r}, t)$ is the fluctuating part of the pressure at \mathbf{r} and t , and the angle brackets denote the ensemble average (mean value). For the sake of simplicity, we will only calculate an off-diagonal element

$$\rho_0^{-1}\langle p(\partial u_z/\partial x + \partial u_x/\partial z) \rangle.$$

We shall first calculate $\langle p\partial u_z/\partial x \rangle$. Afterwards, it will be very simple to calculate the transposed correlation $\langle p\partial u_x/\partial z \rangle$ as well.

To evaluate $\langle p\partial u_z/\partial x \rangle$ we need expressions for p and \mathbf{u} , and both quantities can be obtained from the Navier-Stokes equation. The fluctuating part of that equation is given by

$$\partial\mathbf{u}/\partial t + (\mathbf{u} + \mathbf{U}) \cdot \nabla\mathbf{u} = \langle \mathbf{u} \cdot \nabla\mathbf{u} \rangle - \nabla p/\rho_0 + \mathbf{u} \cdot \nabla\mathbf{U} + \nu\nabla^2\mathbf{u}, \quad (2)$$

where ν is the molecular viscosity, ρ_0 is the fluid density, assumed to be constant, and \mathbf{U} is the mean flow. Equation (2) is obtained from the Navier-Stokes equation by subtracting out the average of the latter.

A formal expression for p is obtained by taking the divergence of (2) and using incompressibility $\nabla \cdot \mathbf{u} = 0$:

$$\frac{\nabla^2 p(t)}{\rho_0} = -\nabla \cdot (\mathbf{u} \cdot \nabla\mathbf{u})' - 2\left(\frac{\partial u_z}{\partial x} \frac{\partial U_0}{\partial z}\right), \quad (3)$$

where we have defined

$$(\mathbf{u} \cdot \nabla\mathbf{u})' \equiv \mathbf{u} \cdot \nabla\mathbf{u} - \langle \mathbf{u} \cdot \nabla\mathbf{u} \rangle \quad (4)$$

to be the fluctuating part of $\mathbf{u} \cdot \nabla \mathbf{u}$, and we have used the idealized flow $\mathbf{U} = [U_0(z), 0, 0]$ so that

$$\nabla \cdot (\mathbf{U} \cdot \nabla \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{U}) = 2 \frac{\partial u_z}{\partial x} \frac{\partial U_0}{\partial z}. \quad (5)$$

Equation (3) can be solved for p by Fourier transforms. The transforms of p and \mathbf{u} are defined by

$$p_{\mathbf{k}}(t) \equiv \int d\mathbf{r} p(t) \exp(-i\mathbf{k} \cdot \mathbf{r}), \quad \mathbf{u}_{\mathbf{k}}(t) \equiv \int d\mathbf{r} \mathbf{u}(\mathbf{r}, t) \exp(-i\mathbf{k} \cdot \mathbf{r}). \quad (6)$$

We then obtain $p_{\mathbf{k}}$ from the Fourier transform of (3):

$$\rho_0^{-1} p_{\mathbf{k}}(t) = N_{\mathbf{k}}(t) + (2ik_x/k^2) u_{z\mathbf{k}}(t) \partial U_0/\partial z, \quad (7)$$

where $N_{\mathbf{k}}(t)$, the Fourier transform of the nonlinear fluctuation term $\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})'$, is explicitly given by

$$N_{\mathbf{k}}(t) \equiv - \int \frac{d\mathbf{k}_1}{(2\pi)^3} \frac{\mathbf{k}\mathbf{k}}{k^2} : [\mathbf{u}_{\mathbf{k}_1}(t) \mathbf{u}_{\mathbf{k}-\mathbf{k}_1}(t) - \langle \mathbf{u}_{\mathbf{k}_1}(t) \mathbf{u}_{\mathbf{k}-\mathbf{k}_1}(t) \rangle], \quad (8)$$

and we have used the inverse Fourier transform

$$u(\mathbf{r}, t) = (2\pi)^{-3} \int d\mathbf{k}_1 \mathbf{u}_{\mathbf{k}_1}(t) \exp(i\mathbf{k}_1 \cdot \mathbf{r})$$

in (8). To obtain (7), we have also neglected the spatial variation of $\partial U_0/\partial z$ as consistent with our limitation of slow variation of mean quantities on the scale of L_0 , the integral wavelength of the spectrum.

It is very convenient to continue the analysis with Fourier transforms. Let us then Fourier expand the pressure-velocity term:

$$\langle p(t) \partial u_z(t)/\partial x \rangle = - \frac{1}{(2\pi)^3 V} \int d\mathbf{k} \langle u_{z\mathbf{k}}^*(t) ik_x p_{\mathbf{k}}(t) \rangle, \quad (9)$$

where V is the volume of the system, $u_{z\mathbf{k}}(t)$ is the z component of $\mathbf{u}_{\mathbf{k}}(t)$,

$$u_{z\mathbf{k}}(t) \equiv \int d\mathbf{r} u_z(t) \exp(-i\mathbf{k} \cdot \mathbf{r}), \quad (10)$$

and the asterisk denotes the complex conjugate. The value of V will not be needed in our final expressions given in terms of the spatial variable \mathbf{r} . We have also used, in (9), the homogeneity condition

$$\langle \mathbf{u}_{z\mathbf{k}_1}^* p_{\mathbf{k}_2} \rangle = \frac{(2\pi)^3}{V} \delta(\mathbf{k}_1 - \mathbf{k}_2) \langle u_{z\mathbf{k}_1}^* p_{\mathbf{k}_1} \rangle,$$

where δ is the Dirac delta function and V is the volume of the system. The validity of this condition is consistent with our limitation of slow spatial variations of average quantities.

Substitution of (7) and (8) in (9) yields

$$\begin{aligned} \rho_0^{-1} \langle p(t) \partial u_z(t) / \partial x \rangle &= A_{xz}^N + A_{xz}^M, \tag{11} \\ A_{xz}^N &\equiv -\frac{i}{(2\pi)^3 V} \int d\mathbf{k} k_x \langle u_{z\mathbf{k}}^*(t) N_{\mathbf{k}}(t) \rangle, \\ A_{xz}^M &\equiv \frac{2}{(2\pi)^3 V} \int d\mathbf{k} \left(\frac{k_x^2}{k^2} \right) \langle u_{z\mathbf{k}}^*(t) u_{z\mathbf{k}}(t) \rangle \frac{\partial U_0}{\partial z}, \end{aligned}$$

where A_{xz}^N is seen to be the contribution to the pressure-strain rate from the non-linear fluctuation part of p , and A_{xz}^M is the contribution from the mean-flow part of p .

It is the term A_{xz}^N for which Rotta (1951) proposed a model that has been widely used. Evaluating this term presents the familiar closure problem of calculating third-order velocity correlations, $\langle u_{z\mathbf{k}}^* N_{\mathbf{k}} \rangle$, in terms of covariances (Reynolds stresses). We will obtain closure for $\langle u_{z\mathbf{k}}^* N_{\mathbf{k}} \rangle$, by first expressing \mathbf{u} in terms of second-order (non-linear) velocity fluctuations so that $\langle u_{z\mathbf{k}}^* N_{\mathbf{k}} \rangle$ can be expressed as a fourth-order velocity correlation. Closure approximations can be applied directly to that fourth-order correlation. The main task is to derive a nonlinear expression for \mathbf{u} . This is done next.

3. Expression for $u_z(t)$

An expression for u_z can be readily obtained by a formal solution (an integration) of (2). It is convenient to first write (2) in the form

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) \mathbf{u}(t) &= \mathbf{I}_\nu(\mathbf{r}, t), \tag{12} \\ \mathbf{I}_\nu(\mathbf{r}, t) &\equiv -[\mathbf{u}(t) \cdot \nabla \mathbf{u}(t)]' - \mathbf{u} \cdot \nabla \mathbf{U} - \frac{\nabla p(t)}{\rho_0} + \nu \nabla^2 \mathbf{u}(t). \end{aligned}$$

A formal solution of (12) can be immediately written as

$$\mathbf{u}(t) = \int d\mathbf{r}_1 G_0(\mathbf{r}, t; \mathbf{r}_1, 0) \mathbf{u}(\mathbf{r}_1, 0) + \int_0^t dt_1 \int d\mathbf{r}_1 G_0(\mathbf{r}, t; \mathbf{r}_1, t_1) \mathbf{I}_\nu(\mathbf{r}_1, t_1) \tag{13}$$

where $G_0(\mathbf{r}, t; \mathbf{r}_1, t_1)$ is a Green's function defined by

$$\left. \begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla \right) G_0(\mathbf{r}, t; \mathbf{r}_1, t_1) &= 0, \\ G_0(\mathbf{r}, t_1; \mathbf{r}_1, t_1) &= \delta(\mathbf{r} - \mathbf{r}_1). \end{aligned} \right\} \tag{14}$$

Equation (13) can be verified by differentiating both sides of (13) with respect to t and substituting (14) and (13). The quantity G_0 can also be viewed as the integrating factor of (12). That is, (12) can be viewed as a linear inhomogeneous equation with the homogeneous part given by its left side, and with G_0 is the solution of that part. Note that $G_0(t)$ is explicitly given by

$$\begin{aligned} G_0 &= \exp [-(t-t_1) \mathbf{U} \cdot \nabla] \delta(\mathbf{r} - \mathbf{r}_1) \\ &= \delta[\mathbf{r} - (t-t_1) \mathbf{U} - \mathbf{r}_1] \end{aligned} \tag{15}$$

for our case in which \mathbf{U} varies slowly, or not at all, with t . More generally, when $\mathbf{U} = \mathbf{U}(t)$ varies with time,

$$G_0 = \exp \left[- \int_0^t dt_2 \mathbf{U}(t_2) \cdot \nabla \right] \delta(\mathbf{r} - \mathbf{r}_1).$$

A simplification we anticipate is that G_0 will be found to drop out of the term $\langle p \partial u_z / \partial x \rangle$.

The z component of (13) gives $u_z(t) \equiv u_z(\mathbf{r}, t)$, which we need for A_{xz}^N , as:

$$u_z(t) = \int d\mathbf{r}_1 G_0(\mathbf{r}, t; \mathbf{r}_1, 0) u_z(\mathbf{r}_1, 0) - \int_0^t dt_1 \int d\mathbf{r}_1 G_0(\mathbf{r}, t; \mathbf{r}_1, t_1) \left[(\mathbf{u} \cdot \nabla u_z)' + \rho_0^{-1} \frac{\partial p}{\partial z_1} - \nu \nabla_1^2 u_z \right]_{t_1, \mathbf{r}_1}, \quad (16)$$

where the subscript t_1, \mathbf{r}_1 on the square bracket is a reminder that the functions inside are evaluated at t_1 and \mathbf{r}_1 [e.g. $p = p(\mathbf{r}_1, t_1)$ and $\mathbf{u} = \mathbf{u}(\mathbf{r}_1, t_1)$ in (16)], z_1 denotes the z component of $\mathbf{r}_1 = (x_1, y_1, z_1)$, ∇_1 denotes $\partial / \partial \mathbf{r}_1$, and we have used

$$(\mathbf{u} \cdot \nabla u_z)' \equiv \mathbf{u} \cdot \nabla u_z - \langle \mathbf{u} \cdot \nabla u_z \rangle.$$

For present simplification we consider large Reynolds number so that $\nu \nabla_1^2 u_z$ can be neglected in (16). This neglect is justified because the scales that are small enough to be influenced by the viscous term do not contribute significantly to the pressure-strain-rate covariance [i.e. such scales do not contribute significantly to the integral in (33) or (37)]. Smaller Reynolds number will be considered in appendix D. Equation (16) thus becomes

$$u_z(t) = \int d\mathbf{r}_1 G_0(\mathbf{r}, t; \mathbf{r}_1, 0) u_z(\mathbf{r}_1, 0) - \int_0^t dt_1 \int d\mathbf{r}_1 G_0(\mathbf{r}, t; \mathbf{r}_1, t_1) \left[(\mathbf{u} \cdot \nabla u_z)' + \rho_0^{-1} \frac{\partial p}{\partial z_1} \right]_{t_1, \mathbf{r}_1}. \quad (17)$$

The Fourier transform of (17), needed for (11), is

$$u_{z\mathbf{k}}(t) = G_{0\mathbf{k}}(t) u_{z\mathbf{k}}(0) - \int_0^t dt_1 G_{0\mathbf{k}}(t - t_1) [(\mathbf{u} \cdot \nabla u_z)'_{\mathbf{k}} + \rho_0^{-1} i k_z p_{\mathbf{k}}]_{t_1}, \quad (18)$$

where

$$(\mathbf{u} \cdot \nabla u_z)'_{\mathbf{k}} \equiv \frac{i}{(2\pi)^3} \int d\mathbf{k}_1 \mathbf{k} \cdot (\mathbf{u}_{\mathbf{k}_1} u_{z\mathbf{k}_2} - \langle \mathbf{u}_{\mathbf{k}_1} u_{z\mathbf{k}_2} \rangle) \quad (19)$$

is the Fourier transform of $(\mathbf{u} \cdot \nabla u_z)'$, $\mathbf{k}_2 \equiv \mathbf{k} - \mathbf{k}_1$, and

$$G_{0\mathbf{k}}(t - t_1) \equiv \int d(\mathbf{r} - \mathbf{r}_1) G_0(\mathbf{r}, t; \mathbf{r}_1, t_1) \exp[-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_1)] \quad (20a)$$

$$\approx \exp(-i\mathbf{k} \cdot \mathbf{U}). \quad (20b)$$

is the Fourier transform of G_0 . For the last step, we used (15).

To eliminate $p_{\mathbf{k}}$ from (18) we substitute (7) and obtain $u_{z\mathbf{k}}$ in the desired form

$$u_{z\mathbf{k}}(t) = G_{0\mathbf{k}}(t) u_{z\mathbf{k}}(0) - \int_0^t dt_1 G_{0\mathbf{k}}(t - t_1) \left[(\mathbf{u} \cdot \nabla u_z)'_{\mathbf{k}} + i k_z N_{\mathbf{k}} - 2 \left(\frac{k_x k_z}{k^2} \right) u_{z\mathbf{k}} \frac{\partial U_0}{\partial z} \right]_{t_1}. \quad (21)$$

Equation (21) determines $u_{z\mathbf{k}}$ in terms of the nonlinear velocity fluctuation $(\mathbf{u} \cdot \nabla u_z)_{\mathbf{k}} + i k_z N_{\mathbf{k}}$. We can use this equation to calculate $A_{xz}^N(t)$.

4. Calculation of A_{xz}^N (nonlinear part of the pressure-strain)

With $u_{z\mathbf{k}}$ given by (21), we can express the triple velocity correlation $\langle u_{z\mathbf{k}}^* N_{\mathbf{k}} \rangle$ and A_{xz}^N in terms of fourth-order velocity correlations.

The fourth-order correlation is obtained by substituting (21) in $\langle u_{z\mathbf{k}}^*(t) N_{\mathbf{k}}(t) \rangle$:

$$\begin{aligned} \langle u_{z\mathbf{k}}^*(t) N_{\mathbf{k}}(t) \rangle &= G_{0\mathbf{k}}^*(t) \langle u_{z\mathbf{k}}^*(0) N_{\mathbf{k}}(t) \rangle - \int_0^t dt_1 G_{0\mathbf{k}}^*(t-t_1) \left\{ \langle [\mathbf{u}(t_1) \cdot \nabla u_z(t_1)]_{\mathbf{k}}^* N_{\mathbf{k}}(t) \rangle \right. \\ &\quad \left. + ik_z \langle N_{\mathbf{k}}^*(t_1) N_{\mathbf{k}}(t) \rangle - \frac{2k_x k_z}{k^2} \langle u_{z\mathbf{k}}^*(t_1) N_{\mathbf{k}}(t) \rangle \frac{\partial U_0}{\partial z} \right\}. \end{aligned} \quad (22)$$

This equation can be substantially simplified at large t (i.e. t larger than the integral time scales) by the following considerations: First, the initial-value term $\langle u_{z\mathbf{k}}^*(0) N_{\mathbf{k}}(t) \rangle$ decays towards zero as t increases,

$$\langle u_{z\mathbf{k}}^*(0) N_{\mathbf{k}}(t) \rangle / \langle u_{z\mathbf{k}}^*(t) N_{\mathbf{k}}(t) \rangle \rightarrow 0 \quad (t \rightarrow \infty), \quad (23)$$

even if the energy increases with t (as in Harris, Graham & Corrsin 1977). It can be shown that the time scale (e -folding time) of this decay is on the order of $(k\nu_0)^{-1}$, where $\frac{3}{2}v_0^2 \equiv \frac{1}{2}\langle \mathbf{u} \cdot \mathbf{u} \rangle$ is the mean-square kinetic energy density of the turbulence. This scale is of the same order as the (Eulerian) decay time of the more familiar velocity covariance $\langle u_{z\mathbf{k}}^*(0) u_{z\mathbf{k}}(t) \rangle$ (e.g. Kraichnan 1959). Secondly, it can be shown, as a corollary to (23), that at large t (compared to integral time scales)

$$\int_0^t dt_1 G_{0\mathbf{k}}^*(t-t_1) \langle u_{z\mathbf{k}}^*(t_1) N_{\mathbf{k}}(t) \rangle = \tau_{\mathbf{k}} \langle u_{z\mathbf{k}}^*(t) N_{\mathbf{k}}(t) \rangle, \quad (24)$$

where $\tau_{\mathbf{k}}$ is also an (Eulerian) integral time scale on the order of $(k\nu_0)^{-1}$. Substituting (23) and (24) into (22), there results (for t larger than integral time scales)

$$\begin{aligned} \langle u_{z\mathbf{k}}^*(t) N_{\mathbf{k}}(t) \rangle &= (1+a)^{-1} \int_0^t dt_1 G_{0\mathbf{k}}(t-t_1) F_{\mathbf{k}}, \\ F_{\mathbf{k}} &\equiv \langle [\mathbf{u}(t_1) \cdot \nabla u_z(t_1)]_{\mathbf{k}}^* N_{\mathbf{k}}(t) \rangle + ik_z \langle N_{\mathbf{k}}^*(t_1) N_{\mathbf{k}}(t) \rangle, \\ a &\equiv -2\tau_{\mathbf{k}} \left(\frac{k_x k_z}{k^2} \right) \frac{\partial U_0}{\partial z}, \end{aligned} \quad (25)$$

which expresses $\langle u_{z\mathbf{k}}^* N_{\mathbf{k}} \rangle$ in terms of the fourth-order velocity correlation denoted by $F_{\mathbf{k}}$. The large t limit is consistent with our basic limitation of small variations on the integral time scale.

The nonlinear part of the pressure-strain, denoted by A_{xz}^N in (11), is expressed in terms of fourth-order correlations by substituting (25) into (11):

$$A_{xz}^N = \frac{i}{(2\pi)^3 V} \int d\mathbf{k} k_x \int_0^t dt_1 G_{0\mathbf{k}}(t-t_1) F_{\mathbf{k}} (1+a)^{-1}. \quad (26)$$

To complete the closure problem for $A_{xz}^N(t)$ we must express $F_{\mathbf{k}}$ in terms of stress spectra (velocity covariances) and then perform the t_1 and \mathbf{k} integrations. An expression for $F_{\mathbf{k}}$ in terms of the stress spectra is derived in appendix A. This derivation is based on the neglect of the cumulant of the two-time fourth-order correlation, $\langle \mathbf{u}_{\mathbf{k}_1}^*(t_1) u_{\mathbf{k}_2}^*(t_1) [\mathbf{u}_{\mathbf{k}_3}(t) \mathbf{u}_{\mathbf{k}_4}(t)]' \rangle$, which occurs in $F_{\mathbf{k}}$. Such a cumulant neglect is also

basic to the direct interaction approximation (e.g. Kraichnan 1959; Leslie 1973; Herring 1966; Edwards 1964; Weinstock 1977), and is much weaker than neglect of single-time fourth-order cumulants in quasi-normal theory (Orzag 1970; Proudman & Reid 1954). The closure expression thereby obtained for A_{xz}^N in appendix A is

$$A_{xz}^N = -2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{(2\pi)^3} \frac{\tau_c \mathbf{b}(\mathbf{k})}{1+a} : \mathbf{S}(\mathbf{k}_2) \mathbf{S}(\mathbf{k}_1) : \frac{\mathbf{k}^2}{k^2}, \quad (27)$$

where

$$\mathbf{S}(\mathbf{k}) \equiv \langle \mathbf{u}(\mathbf{k}, t) \mathbf{u}^*(\mathbf{k}, t) \rangle V^{-1}, \quad \int \frac{d\mathbf{k}}{(2\pi)^3} \mathbf{S}(\mathbf{k}) = \langle \mathbf{u} \mathbf{u} \rangle$$

is the velocity covariance spectrum at wave vector \mathbf{k} and time t , $\mathbf{k}_2 \equiv \mathbf{k} - \mathbf{k}_1$, and $\mathbf{b}(\mathbf{k})$ and τ_c (an Eulerian correlation time, or damping time) are defined by

$$\mathbf{b}(\mathbf{k}) \equiv k_x \mathbf{k} \hat{z} - k_x k_z \mathbf{k}^2 / k^2, \\ \tau_c \equiv (\frac{1}{2}\pi)^{\frac{1}{2}} [(\mathbf{k}_1^2 + \mathbf{k}_2^2) : \langle \mathbf{u} \mathbf{u} \rangle]^{-\frac{1}{2}}.$$

Equation (27) determines the pressure-strain rate A_{xz}^N in terms of measurable quantities (velocity spectra \mathbf{S}). This equation is a principal result of our paper. If the spectra \mathbf{S} were known, by theory or experiment, then it would be straightforward to evaluate the \mathbf{k} and \mathbf{k}_2 integrals. At present, some aspects of \mathbf{S} are known fairly well and some are not. Hence, presently, (27) can only be evaluated for models of parts of \mathbf{S} , or by making assumptions about \mathbf{S} . Afterwards, one can readily determine the sensitivity of A_{xz}^N to these models.

Note that no approximations have been made about the anisotropy so that (27) is correct to all orders (if expanded in powers of the anisotropy). Note, too, that A_{xz}^N vanishes for isotropic turbulence (i.e. for isotropic \mathbf{S} , $\langle \mathbf{u} \mathbf{u} \rangle$, and vanishing $\partial U_0 / \partial z$) because, then, the integrand is an odd function of k_x . The main limitation of (27) is to slow variations of mean quantities in space and time. The main approximation is the cumulant discard discussed in appendix A.

In order to derive a Rotta-like expression from (27), we must linearize it in terms of the anisotropy. This is done next.

5. A_{xz}^N to first order in anisotropy (the Rotta term)

The purpose of this section is to explicitly express A_{xz}^N in terms of the Reynolds stress. This expression is only derived to first order in the anisotropy, and a comparison is made with Rotta's model. Afterwards, the theoretical value of Rotta's constant is calculated (approximately) – in § 6.

To expand A_{xz}^N in powers of anisotropy, we divide $\mathbf{S}(\mathbf{k})$, in (27), into an isotropic part $\mathbf{S}(\mathbf{k})^I$ and an anisotropic deviation $\mathbf{S}(\mathbf{k})^A$ as follows:

$$\left. \begin{aligned} \mathbf{S}(\mathbf{k}) &\equiv \mathbf{S}(\mathbf{k})^I + \mathbf{S}(\mathbf{k})^A, \\ \mathbf{S}(\mathbf{k})^I &\equiv 2\pi^2 \left(\mathbf{I} - \frac{\mathbf{k}\mathbf{k}}{k^2} \right) \frac{E(k)}{k^2}, \end{aligned} \right\} \quad (28)$$

where \mathbf{I} is the identity matrix, and $E(k)$ is a scalar energy spectrum. It satisfies

$$\int_0^\infty dk E(k) = \frac{1}{2} \langle \mathbf{u} \cdot \mathbf{u} \rangle \equiv \frac{3}{2} v_0^2. \quad (28')$$

Similarly, the stress tensor can be divided into an isotropic part $v_0^2 \mathbf{l}$ and an anisotropic part $\langle \mathbf{uu} \rangle^A$

$$\langle \mathbf{uu} \rangle = v_0^2 \mathbf{l} + \langle \mathbf{uu} \rangle^A. \quad (29)$$

Equation (27) can now be linearized by substituting (28) and (29), and neglecting all second- and higher-order terms in the anisotropies \mathbf{S}^A and $\langle \mathbf{uu} \rangle^A$. This neglect yields

$$A_{xz}^N = -2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \left\{ \frac{\mathbf{b}(\mathbf{k}) : [\mathbf{S}(\mathbf{k}_1) \mathbf{S}^I(\mathbf{k}_2) + \mathbf{S}^I(\mathbf{k}_1) \mathbf{S}(\mathbf{k}_2)] : \mathbf{k}^2}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2 v_0} \right. \\ \left. - \frac{1}{2} \frac{\mathbf{b}(\mathbf{k}) : \mathbf{S}^I(\mathbf{k}_1) \mathbf{S}^I(\mathbf{k}_2) : \mathbf{k}^2 [(\mathbf{k}_1^2 + \mathbf{k}_2^2) : \langle \mathbf{uu} \rangle^A]}{(k_1^2 + k_2^2)^{\frac{3}{2}} v_0^3 k^2} \right\}, \quad (30)$$

which gives A_{xz}^N to first order in the anisotropy.

It is not difficult to express the right side of (30) in terms of the Reynolds stress. A useful simplification comes from incompressibility:

$$\mathbf{k}_2 \cdot \mathbf{S}(\mathbf{k}_2) = 0, \quad \mathbf{k} \cdot \mathbf{S}(\mathbf{k}_2) = (\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{S}(\mathbf{k}_2) = \mathbf{k}_1 \cdot \mathbf{S}(\mathbf{k}_2).$$

The first term in the integrand of (30) can thus be expressed, with (28), as

$$\mathbf{b}(\mathbf{k}) : \mathbf{S}(\mathbf{k}_1) \mathbf{S}^I(\mathbf{k}_2) : \mathbf{k}^2 k^{-2} = \left[k_x (\mathbf{k}_2 \cdot \mathbf{S}(\mathbf{k}_1) \cdot \hat{\mathbf{z}}) - \left(\frac{k_x k_z}{k^2} \right) (\mathbf{k}_2 \cdot \mathbf{S}(\mathbf{k}_1) \cdot \mathbf{k}_2) \right] \\ \times \left[k_1^2 - \frac{(\mathbf{k}_2 \cdot \mathbf{k}_1)^2}{k_2^2} \right] \frac{2\pi^2 E(k_2)}{k_2 k_2^2}, \quad (31)$$

and the second term is expressed, after interchanging the dummy variables of integration ($\mathbf{k}_1 \rightarrow \mathbf{k}_2$, $\mathbf{k}_2 \rightarrow \mathbf{k}_1$), as

$$\mathbf{b}(\mathbf{k}) : \mathbf{S}^I(\mathbf{k}_2) \mathbf{S}(\mathbf{k}_1) : \mathbf{k}^2 k^{-2} = \left[k_x \left(k_{1z} - \frac{k_{2z}(\mathbf{k}_2 \cdot \mathbf{k}_1)}{k_2^2} \right) - \frac{k_x k_z}{k^2} \left(k_1^2 - \frac{(\mathbf{k}_2 \cdot \mathbf{k}_1)^2}{k_2^2} \right) \right] \\ \times (\mathbf{k}_2 \cdot \mathbf{S}(\mathbf{k}_1) \cdot \mathbf{k}_2) \frac{2\pi^2 E(k_2)}{k^2 k_2^2}. \quad (32)$$

The third term of (30) can be evaluated in a straightforward (although lengthy) integration, and we have found it very small in comparison with (31); it is henceforth neglected.

We wish to separate $S_{zz}(\mathbf{k}_1) \equiv \hat{\mathbf{x}} \cdot \mathbf{S} \cdot \hat{\mathbf{z}}$ from all the other spectral elements [$S_{xx}(\mathbf{k}_1)$, $S_{yy}(\mathbf{k}_1)$, $S_{zz}(\mathbf{k}_1)$, $S_{xy}(\mathbf{k}_1)$, $S_{yz}(\mathbf{k}_1)$] in (31) and (32). The reason behind this separation, as will soon be made clear, is that the Rotta hypothesis predicts that A_{xz}^N should depend on S_{xz} in the form $\int d\mathbf{k}_1 S_{xz}(\mathbf{k}_1)$ and not on the other spectral elements (e.g. S_{xx}). In fact, if the Rotta hypothesis were exact, then all the spectral elements other than S_{xz} would cancel out of (30). To effect this cancellation, it is necessary to express S_{xy} and S_{yz} in terms of S_{xz} by means of the incompressibility conditions

$$\mathbf{k}_1 \cdot \mathbf{S}(\mathbf{k}_1) \cdot \hat{\mathbf{x}} \equiv k_{1x} S_{xx}(\mathbf{k}_1) + k_{1y} S_{xy}(\mathbf{k}_1) + k_{1z} S_{xz}(\mathbf{k}_1) = 0,$$

$$\mathbf{k}_1 \cdot \mathbf{S}(\mathbf{k}_1) \cdot \hat{\mathbf{z}} \equiv k_{1x} S_{xz}(\mathbf{k}_1) + k_{1y} S_{yz}(\mathbf{k}_1) + k_{1z} S_{zz}(\mathbf{k}_1) = 0.$$

These expressions are substituted for S_{xy} and S_{yz} in (31) and (32), and, for later convenience (see §7e), we also insert the expression

$$\Delta \equiv \frac{k_{1x}k_{1z}}{k^2} \left[\left(\frac{2k_{2x}k_{2y}}{k_{1y}} \right) \mathbf{k}_1 \cdot \mathbf{S}(\mathbf{k}_1) \cdot \hat{\mathbf{x}} + \left(\frac{2k_{2y}k_{2z}}{k_{1y}} \right) \mathbf{k}_1 \cdot \mathbf{S}(\mathbf{k}_1) \cdot \hat{\mathbf{z}} \right] [1 - \frac{1}{2}(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2] = 0$$

in (31) and (32). After these substitutions are made in (31) and (32) we obtain the temporarily lengthy expressions

$$\begin{aligned} \frac{\mathbf{b}(\mathbf{k}) : \mathbf{S}(\mathbf{k}_1) \mathbf{S}^I(\mathbf{k}_2) : \mathbf{k}^2}{k^2} &= \left\{ k_x \left[k_{2x} S_{xz} - \frac{k_{2y}}{k_{1y}} (k_{1x} S_{xz} + k_{1z} S_{zz}) + k_{2z} S_{zz} \right] \right. \\ &\quad - \frac{k_x k_z}{k^2} \left[k_{2x}^2 S_{xx} + k_{2y}^2 S_{yy} + k_{2z}^2 S_{zz} - \frac{2k_{2x}k_{2y}}{k_{1y}} (k_{1z} S_{xz} + k_{1x} S_{xx}) \right. \\ &\quad \left. \left. - \frac{2k_{2y}k_{2z}}{k_{1y}} (k_{1x} S_{xz} + k_{1z} S_{zz}) + 2k_{2x}k_{2z} S_{xz} \right] - 2\Delta \right\} \\ &\quad \times \frac{2\pi^2 E(k_2)}{k^2 k_2^2} [k_1^2 - (\mathbf{k}_1 \cdot \hat{\mathbf{k}}_2)^2] \end{aligned} \quad (31')$$

and

$$\begin{aligned} \frac{\mathbf{b}(\mathbf{k}) : \mathbf{S}^I(\mathbf{k}_2) \mathbf{S}(\mathbf{k}_1) : \mathbf{k}^2}{k^2} &= \left\{ \left[k_x \left(k_{1z} - \frac{k_{2z}(\hat{\mathbf{k}}_2 \cdot \mathbf{k}_1)}{k_2} \right) - \frac{k_x k_z}{k^2} (k_1^2 - (\hat{\mathbf{k}}_2 \cdot \mathbf{k}_1)^2) \right] \right. \\ &\quad \times \left[k_{2x}^2 S_{xx} + k_{2y}^2 S_{yy} + k_{2z}^2 S_{zz} - \frac{2k_{2x}k_{2y}}{k_{1y}} (k_{1z} S_{xz} + k_{1x} S_{xx}) \right. \\ &\quad \left. \left. - \frac{2k_{2y}k_{2z}}{k_{1y}} (k_{1x} S_{xz} + k_{1z} S_{zz}) + 2k_{2x}k_{2z} S_{xz} \right] + \Delta \right\} \frac{2\pi^2 E(k)}{k^2 k_2^2}, \end{aligned} \quad (32')$$

where $\hat{\mathbf{k}}_2 \equiv \mathbf{k}_2/k_2$ denotes the unit vector along \mathbf{k}_2 , and to condense notation we use S_{ij} to denote $S_{ij}(\mathbf{k}_1)$ so that, for example, $S_{xz} \equiv S_{xz}(\mathbf{k}_1)$.

To express A_{xz}^N in terms of S_{xz} , we next substitute (31') and (32') back into (30) and divide the result in two parts: one part containing only the spectrum S_{xz} and a remainder part, denoted by R' , containing all the other spectra S_{xx} , S_{zz} , S_{yy} , S_{xy} , and S_{yz} . Thus, we write (30)

$$A_{xz}^N = -2 \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{\gamma E(k_2) S_{xz}(\mathbf{k}_1)}{v_0 k_2^2} + R', \quad (33)$$

where

$$\begin{aligned} \gamma &\equiv \left(k_x k_{2x} - \frac{k_x k_{1x} k_{2y}}{k_{1y}} \right) \frac{k_1^2 - (\mathbf{k}_1 \cdot \hat{\mathbf{k}}_2)^2}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2} - \left\{ \frac{2k_x k_z}{k^2} [k_1^2 - (\mathbf{k}_1 \cdot \hat{\mathbf{k}}_2)^2] \right. \\ &\quad \left. - k_x \left[k_{1z} - \frac{k_{2z}(\mathbf{k}_1 \cdot \hat{\mathbf{k}}_2)}{k_2} \right] \right\} \left\{ \frac{2k_{2x}k_{2z} - 2k_{2y}k_{1y}^{-1}(k_{2x}k_{1z} + k_{1x}k_{2z})}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2} \right\} \\ &\quad - \frac{k_{1x}k_{1z}}{k^2} \left\{ \frac{2[k_1^2 - (\mathbf{k}_1 \cdot \hat{\mathbf{k}}_2)^2]}{k_2} - 1 \right\} \frac{2k_{2y}k_{1y}^{-1}(k_{2x}k_{1z} + k_{1x}k_{2z})}{(k_1^2 + k_2^2)^{\frac{1}{2}}} [1 - \frac{1}{2}(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2], \end{aligned}$$

where R' denotes all the terms coming from (31') and (32') that contain spectra other than S_{xz} . The prime on R' is a reminder that it (R') does not contain S_{xx} . Note that the \mathbf{k} integration has been transformed into a \mathbf{k}_2 integration by $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$. We also wish to reassure the reader that, although γ is quite complex looking, the integrations

in (33) can be easily performed, as is done later on. Furthermore, the first term (containing $k_x k_{2x}$) is dominant; the other terms of γ are small and provide a small correction.

Equation (33) can be easily written in terms of the stress. To do so we simply multiply the numerator and denominator of the S_{xz} part of (33) by $\langle u_x u_z \rangle$, and use the definitions $(2\pi)^{-3} \int d\mathbf{k} S_{xz}(\mathbf{k}) = \langle u_x u_z \rangle$ and $(4\pi)^{-1} \int d\mathbf{k}_2 E(k_2) k_2^{-2} = \frac{3}{2} v_0^2$ to obtain

$$A_{xz}^N = -k^* v_0 \langle u_x u_z \rangle + R', \tag{34}$$

where k^* is a wavenumber explicitly defined by

$$k^* \equiv 3 \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \frac{d\mathbf{k}_2}{4\pi} \frac{\gamma E(k_2) S_{xz}(\mathbf{k}_1)}{k_2^2} \left[\int \frac{d\mathbf{k}_1}{(2\pi)^3} S_{xz}(\mathbf{k}_1) \int \frac{d\mathbf{k}_2}{4\pi} \frac{E(k_2)}{k_2^2} \right]^{-1}, \tag{34'}$$

so that k^* is the mean value of γ averaged over the velocity spectra $S_{xz}(\mathbf{k}_1)$ and $E(k_2) k_2^{-2}$. The quantity $(k^*)^{-1}$ is a novel kind of integral scale because it is a double integral over two spectra. This integral scale is basic to, and characteristic of, the pressure-strain-rate tensor; a knowledge of k^* is equivalent to a knowledge of A_{xz}^N . It is shown in the next section that k^* is readily calculated.

The first term on the right-hand side of (34) agrees with Rotta's model; we refer to it as the 'Rotta term'. This is the term we are interested in, and to which we devote our attention. The R' term is a deviation from Rotta's model. That is, it can be seen, by using $(2\pi)^{-3} \int d\mathbf{k} S_{ij}(\mathbf{k}) \equiv \langle u_i u_j \rangle$, that R' is linearly proportional to the stresses other than $\langle u_x u_z \rangle$, and, hence, constitutes a departure from Rotta's model. We will carry this term along even though it is found to be very small in §7(e).

Returning our attention to the first term of (34), we note that it is almost in the form suggested by Rotta. There only remains to evaluate k^* in terms of the energy dissipation rate ϵ and the turbulence kinetic energy density $e_0 \equiv \frac{3}{2} v_0^2$. To make this evaluation, we have only to note that $(k^* v_0)^{-1}$ and ϵ/e_0 both have the same dimensions (time) so that (34) can be expressed in the Rotta form

$$A_{xz}^N = -\frac{1}{2} C_{xz} \left(\frac{\epsilon}{e_0} \right) \langle u_x u_z \rangle + R', \tag{35}$$

where C_{xz} is a dimensionless proportionality constant given by

$$C_{xz} \equiv 2k^* v_0 (\epsilon/e_0)^{-1} \tag{36}$$

or, equivalently, with k^* given by (34'),

$$C_{xz} \equiv 6 \left(\frac{\pi}{2} \right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon} \right) \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{\gamma E(k_2) S_{xz}(\mathbf{k}_1)}{k_2^2 e_0 \langle u_x u_z \rangle}. \tag{37}$$

The numerical value of C_{xz} is calculated in §6. Equation (35) determines A_{xz}^N in terms of $\langle u_x u_z \rangle$. However, for a comparison with Rotta's model we need the symmetrized quantity $A_{xz}^N + A_{zx}^N$. The transpose correlation A_{zx}^N is readily obtained by taking the transpose of our previous equations. It is thereby found that $A_{zx}^N = A_{xz}^N$ to first order in the anisotropy. Hence, we finally have, from (35), the desired expression for the (nonlinear part of the) pressure-strain rate in the Rotta form:

$$A_{xz}^N + A_{zx}^N = -C_{xz} \frac{\epsilon}{e_0} \langle u_x u_z \rangle + 2R', \tag{38}$$

The first term on the right agrees with Rotta's model. The term $2R'$ constitutes a small deviation.

Thus, statistical turbulence theory has been used to derive Rotta's model to first order in the anisotropy for an off-diagonal element of $\rho_0^{-1}\langle p[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] \rangle$. This verification is only partial because Rotta's model requires that equations analogous to (38) be valid for all the elements of the pressure-strain tensor, and that the proportionality constant be the same for each element (e.g. $C_{xz} = C_{xx} = C_{zz}$).

An important distinction between (38) and Rotta's model has to do with the dimensionless constant C_{xz} : in the latter, C_{xz} is an undetermined constant that must be estimated empirically. In (38), on the other hand, C_{xz} is explicitly defined by (37) in terms of the (shear stress) spectrum, and can be calculated in principle (see §6). This might have practical importance because it is very difficult to obtain a *direct* experimental measurement of A_{xz}^N or, consequently, of C_{xz} . To our knowledge, such measurements have not been made away from a boundary. Neither have the diagonal elements (e.g. A_{xx}^N) been measured. Note that C_{xz} is actually the ratio of two characteristic decay times, a velocity correlation time and the viscous dissipation rate. Both times are increasing functions of the total turbulence energy, which tends to cancel out, and, consequently, C_{xz} may not be too sensitive to the energy-containing part of the spectrum. This is borne out in (44).

6. Theoretical calculation of C_{xz}

In this section, C_{xz} is calculated by performing the \mathbf{k}_1 and \mathbf{k}_2 integrations in (37). These integrations require that some assumptions or models be used for the spectrum S_{xz} . The sensitivity of C_{xz} to the models will be examined afterwards.

In spherical co-ordinates, the integrations over the directions of \mathbf{k}_1 and \mathbf{k}_2 in (37) can be performed by making straightforward approximations. These integrations are given in appendix B, where it is shown that (37) reduces to

$$C_{xz} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_1^2 k_2^2 E(k_2) E_{xz}(k_1)}{(k_1^2 + k_2^2)^{\frac{3}{2}} e_0 \langle u_x u_z \rangle}. \quad (39)$$

Here, the scalar spectrum $E_{xz}(k_1)$ is the integral of $S_{xz}(\mathbf{k}_1)$ over a spherical shell of radius k_1 ; that is,

$$E_{xz}(k_1) \equiv \frac{k_1^2}{(2\pi)^3} \int_0^{2\pi} d\phi_1 \int_0^\pi d\theta_1 \sin\theta_1 S_{xz}(\mathbf{k}_1), \quad (40)$$

where θ_1 is the angle that \mathbf{k}_1 makes with the \mathbf{x} axis ($k_{1x} = k_1 \cos\theta_1$) and ϕ_1 is the (azimuthal) angle of \mathbf{k}_1 in the plane perpendicular to the \mathbf{x} axis.

This scalar spectrum satisfies

$$\int_0^\infty dk_1 E_{xz}(k_1) \equiv \int \frac{d\mathbf{k}_1}{(2\pi)^3} S_{xz}(\mathbf{k}_1) = \langle u_x u_z \rangle, \quad (40')$$

where $-\langle u_x u_z \rangle$ is the $\hat{\mathbf{x}}\hat{\mathbf{z}}$ element of the stress tensor.

The main approximation was to simplify the integrals of (37) by taking $k_1 \approx k_2$ for some of the factors in γ . This approximation is investigated and justified in §7(b).

To complete the evaluation of C_{xz} , we have only to carry out integrations in (39). This will be done for a class of models of $E(k_2)$ and $E_{xz}(k_1)$, and, afterwards, the sensitivity of C_{xz} to these models is examined.

Preliminary rough estimate of C_{xz}

As a preliminary to the model calculation of C_{xz} , it is enlightening to first quickly estimate it and k^* from a general consideration. To do so, we make the familiar assumption that $E(k)$ has a maximum value, or peak, at some wavenumber k_L , and that the main contribution to the energy integral $\int_0^\infty dk E(k)$ comes from k in the vicinity of k_L . This vicinity is called the energy-containing region. Similarly, we make the additional assumption that the peak of $E_{xz}(k)$ is at or near k_L . In that case the main contribution to the integrations in (39) comes from $k_1 \approx k_L$, $k_2 \approx k_L$, because the factor $k_1^2 k_2^2 (k_1^2 + k_2^2)^{-\frac{3}{2}}$ varies more slowly than $E(k_2) E_{xz}(k_1)$ for most values of k_1 and k_2 . Hence we set $k_1 \approx k_2 \approx k_L$ in that factor to approximate (39) as

$$\begin{aligned} C_{xz} &\sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_L^4}{2^{\frac{3}{2}} k_L^3} \frac{E(k_2) E_{xz}(k_1)}{e_0 \langle u_x u_z \rangle} \\ &\sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \frac{k_L}{2^{\frac{3}{2}}} \int_0^\infty dk_2 \frac{E(k_2)}{e_0} \int_0^\infty dk_1 \frac{E_{xz}(k_1)}{\langle u_x u_z \rangle} \\ &\sim \left(\frac{\pi}{16}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0 k_L}{\epsilon}\right) \end{aligned} \tag{41}$$

(preliminary very rough estimate). The estimate of k^* is obtained by substituting (41) in (36):

$$k^* \sim \left(\frac{\pi}{64}\right)^{\frac{1}{2}} k_L. \tag{42}$$

Equation (41) shows that k^* is related to k_L in a simple way, and is not as obscure as it may have seemed at first glance.

Precise calculation of C_{xz} for model spectra

To obtain a more precise evaluation of C_{xz} and k^* we must resort to a model of $E(k)$ and $E_{xz}(k)$. Afterwards we will examine the sensitivity of k^* and C_{xz} to that model. A convenient model of $E(k)$ was used by Reynolds (1976) to estimate a parameter of decaying turbulence, and was previously used by Comte-Bellot & Corrsin (1966). It is given by $E(k) = \alpha \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ for $k > k_L$ and $E(k) = \alpha \epsilon^{\frac{2}{3}} (k_L^{-\frac{5}{3}-m}) k^m$ for $k \leq k_L$, where $m > -1$ and $\alpha \approx 1.5$ is the Kolgomoroff constant. The spectrum of $E_{xz}(k)$ decreases more rapidly than $E(k)$ as k increases above k_L . For examples, Kaimal *et al.* (1972) find that $S_{xz}(k_x)$, a one-dimensional spectrum closely related to $E_{xz}(k)$, varies as $k_x^{-\frac{7}{3}}$ for $k_x > k_L$, and Panofsky & Mares (1968) find $S_{xz}(k_x) \propto k_x^{-\frac{8}{3}}$. We therefore feel it is reasonable to take $E_{xz}(k) \propto k^{-\frac{7}{3}}$ for $k > k_L$. [The data of Champagne, Harris & Corrsin (1970) also show that $S_{xz}(k_x)$ decreases faster than $k^{-\frac{7}{3}}$ in the inertial sub-range.] Hence, our model $E_{xz}(k)$ is taken to be $E_{xz}(k) = B k^{-\frac{7}{3}}$ for $k \geq k_L$ and $E_{xz}(k) = B (k_L^{-\frac{7}{3}-m}) k^m$ for $k \leq k_L$; where B is a normalizing constant determined to be $B = k_L^{\frac{8}{3}} [\frac{2}{3} + (m+1)^{-1}]^{-1} \langle u_x u_z \rangle$ by requiring that $E_{xz}(k)$ satisfy

$$\int_0^\infty dk E_{xz}(k) \equiv \langle u_x u_z \rangle.$$

Substituting these expressions for E and E_{xx} into (34') and (28') yields k^* and v_0^2 as

$$k^* \approx 0.52 \left(\frac{\pi}{8}\right)^{\frac{1}{2}} [1 + 1.3(m+1)^{-1}]^{-1} k_L, \quad (43a)$$

$$v_0^2 = \alpha \epsilon^{\frac{2}{3}} k^{-\frac{2}{3}} [1 + \frac{2}{3}(m+1)^{-1}] = \frac{2}{3} e_0. \quad (43b)$$

Note that this k^* is quite close to (within 20% of) the general approximation (43) for a wide range of m ($1 \leq m \leq 6$).

Finally, C_{xx} is obtained by substituting the model expressions for E and E_{xx} into (39) or, equivalently, by substituting (43a) and (43b) into (36):

$$C_{xx} = \frac{3}{2} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (0.52) \alpha^{\frac{2}{3}} \frac{[1 + \frac{2}{3}(m+1)^{-1}]^{\frac{2}{3}}}{[1 + 1.3(m+1)^{-1}]},$$

$$\left. \begin{aligned} C_{xx} &= 1.8 \quad (\text{for } m = 0) \\ C_{xx} &= 1.7 \quad (\text{for } m = 4) \end{aligned} \right\} \quad (44)$$

(a small correction is given in (44')), which is seen to be insensitive to m , the large-wavelength behaviour of the spectrum. A small correction discussed next paragraph gives $C_{xx} = 1.6$ to 1.7 [see (44')]. This theoretical value of C_{xx} is fairly close to recent empirical determinations of the Rotta constant (e.g. Launder *et al.* 1975; Reynolds 1976). It exceeds them by only 20%. However, it exceeds by a factor of 1.7 the Rotta constant C that was determined by Lumley & Newman (1977) from the experimental data of Comte-Bellot & Corrsin (1966). That is, their value $C = 2$ corresponds to $C_{xx} = 1$ when our definition is used for the Rotta constant. That value of C is significant because, as was pointed out by Lumley & Newman (1977), it corresponds to no return to isotropy for very weakly anisotropic turbulence at infinite Reynolds number. It is beyond the scope of our paper to discuss the latter phenomenon (i.e. the observations of extremely weak return to isotropy of weakly anisotropic turbulence). Instead, we will mention several considerations, each of which might account for difference between our value of C_{xx} and the Lumley & Newman value of C : The first consideration is that there may be no difference because we calculated the *off-diagonal* element of the pressure-strain-rate tensor whereas Lumley & Newman determined C from the *diagonal* elements. To our knowledge there is no unquestionable proof that, to lowest order in anisotropy, the Rotta constant is exactly the same for all the elements of the pressure-strain rate; that remains to be seen. Furthermore, the off-diagonal elements of the pressure-strain rate and of the stress tensor were probably zero in the Comte-Bellot & Corrsin experiment (they were not measured), and, hence, those elements and C_{xx} may have little to do with the observed slow return to isotropy of the diagonal elements. It should also be noted that recent measurements of fully developed turbulence (Harris *et al.* 1977) show fairly large differences between the various C_{ij} 's. However, that turbulence was appreciably anisotropic and the conclusions therefrom are not applicable to weak anisotropy. The second consideration is that the experimental decay rate was equal to the viscous dissipation rate; a rapid decay which violates our stationarity assumption. Neither (23) nor (24) can be *a priori* used in (22) for such a rapid decay, and, correspondingly, the correlation time τ_c in (A 11) and (27) could be significantly altered, and so could the value of C_{xx} . We wish to add that we know of no theory that is *a priori* quantitatively correct for such a rapid decay. A third consideration is the error caused by the cumulant neglect. This error is

thought to be small, but we really do not know its magnitude. Any of these considerations provides a potential explanation of the difference between our value of C_{xz} and the value of C determined by Lumley & Newman (1977). In fact, each of these considerations point to an area of future investigation:

- (1) the ratios of C_{xz} , C_{xx} , C_{zz} and C_{yy} to each other;
- (2) the influence of rapid time decay on C_{xz} ; and
- (3) the error caused by the cumulant neglect in appendix A.

In view of the number of approximations that have been made about the spectra, it seems surprising at first glance that our theoretical value of C_{xz} is as close as it is to several previous determinations. We believe that this agreement is not entirely fortuitous. To explain why we believe so, we next examine these approximations for the errors they introduce.

7. Errors of theoretical C_{xz} caused by approximations

In this section we wish to examine the approximations or assumptions in our calculation of C_{xz} . These are: (a) the model long-wavelength behaviour of $E(k_2)$ and $E_{xz}(k_1)$; (b) the assumption that the maximum of E_{xz} occurs near the same wavenumber as the maximum of E ; (c) the assumed power laws $E_{xz}(k) \propto k^{-\frac{5}{2}}$ and $E(k) \propto k^{-\frac{5}{2}}$ at large k ; (d) approximations used in appendix B related to assumption (b); (e) the neglect of the remainder term R' ; and (f) the cumulant neglect (appendix A) used to obtain (A 4).

Assumption (a). It is seen in (44) that C_{xz} is, fortunately, quite insensitive to the behaviour of E and E_{xz} at large wavelength. In fact, (44) shows that C_{xz} will only vary by 5% as m varies from 0 to ∞ . This insensitivity is due to a cancellation between the m dependencies of k^* and v_0 .

Assumption (b). To discuss this assumption, let k_L denote the wavenumber where $E(k)$ has its maximum, and let k'_L denote where $E_{xz}(k)$ has its maximum. It is shown in appendix C, for simplified models of $E(k)$ and $E_{xz}(k)$, that, if $k'_L/k_L = 0.5, 0.7, 0.8$ and 2.0 , then C_{xz} would be reduced by 25, 3, 0, and 25% respectively. The maximum value of C_{xz} occurs near $k'_L/k_L = 0.9$. From the data of Champagne *et al.* (1970) and Kaimal *et al.* (1972), it is argued in appendix C that $k'_L \approx 0.7k_L$. In that case, C_{xz} would be reduced by only 3%. However, it is cautioned that this reduction was based on several assumptions that were used to estimate three-dimensional scalar spectra from measurements of one-dimensional spectra. A more precise determination of k'_L/k_L must await future experiments. Nevertheless, it does suggest that C_{xz} could be reduced a little from the value in (44) to

$$\left. \begin{aligned} C_{xz} &= 1.7 \quad (\text{for } m = 0), \\ &= 1.6 \quad (\text{for } m = 4). \end{aligned} \right\} \tag{44'}$$

Assumption (c). The assumed power laws $E(k) \propto k^{-\frac{5}{2}}$ and $E_{xz}(k) \propto k^{-\frac{5}{2}}$ at large k ($k > k_L$) are more reliable than the other assumptions. (We ignore the dissipation range for large Reynolds numbers.) Although these power laws are well founded, we have calculated how deviations in them would alter C_{xz} . These calculations show that C_{xz} changes by less than 5% as the power law of E_{xz} is varied from $k^{-\frac{5}{2}}$ to k^{-2} or k^{-3} at large k . On the other hand, it is found that C_{xz} is very sensitive to the behaviour of $E(k)$ at large k . This is because C_{xz} explicitly depends on e_0/ϵ [see (36)], and the

latter strongly depends on the behaviour of $E(k)$ at large k . Thus, the numerical value of $\epsilon(e_0 v_0 k_L)^{-1}$ given by (43b) is correct if $E(k) \propto k^{-\frac{5}{3}}$ for $k > k_L$. Otherwise, it and C_{xz} could be very different. Our basic assumption for evaluating C_{xz} is that the $-\frac{5}{3}$ power law of $E(k)$ is valid at large Reynolds number. At low or even moderate Reynolds number C_{xz} will differ significantly from (44'). In fact, it is not difficult to show that $(v_0 e_0 k_L)/\epsilon$ and, consequently, C_{xz} decrease by 25% when the Reynolds number decreases to 30 from much larger values – with $E(k) \propto k^{-\frac{5}{3}}$. At constant Reynolds number it is not hard to show, for a power-law spectrum $E(k) \propto k^{-s}$, that the value of $(v_0 e_0 k_L)/\epsilon$ increases as the power s decreases. Hence, C_{xz} will likewise increase as s decreases. [It is interesting to note, from (34), that A_{xz}^N is not very sensitive to the form of $E(k)$, although C_{xz} is. For this reason, it may sometimes be more advantageous to use (34) than (35).]

Assumption (d). Approximations in appendix B that pertain to the spectra include those in (B 5) and the assumption that $2k_1^2(k_1^2 + k_2^2)^{-1}$ can be taken equal to 1 in (B 17): this assumption is examined and accounted for in appendix C and in assumption (b) where we discussed cases of $k'_L \neq k_L$. The assumption in (B 5) is also justifiable by the considerations in appendix C where it is found that $k'_L \approx 0.7k_L$, which implies that $k_1 \approx 0.7k_2$ in (B 5). Hence $2k_1k_2/(k_1^2 + k_2^2) \approx 0.93$. In addition $\cos\theta$ is usually small (i.e. $\frac{1}{2} \int_0^\pi d(\cos\theta) \cos^2\theta = \frac{1}{3}$). Hence the average error (over θ) in (B 5) is about $(1 - 0.93) \times \frac{1}{3} \approx 0.02$.

Assumption (e). The remainder R' was obtained from the substitution of (31') and (32') into (30). It was defined as the collection of all such terms that contain the spectra S_{xx} , S_{yy} , S_{zz} , S_{xy} , or S_{yz} (and not S_{xz}). Hence, R' contains \mathbf{k}_1 and \mathbf{k}_2 integrals of spectra other than S_{xz} . These integrals can be evaluated in the same way as the right-hand side of (37) was evaluated to obtain C_{xz} . We have evaluated R' this way and found it to be very small: only a few percent of A_{xz}^N . That calculation of R' will not be given here because, although straightforward, it is lengthy, and we do not want to enlarge this paper unnecessarily. The purpose of adding $\Delta \equiv 0$ to (31') and (32') can now be explained. The presence of Δ adds S_{xx} and S_{zz} in such a way as to cause R' to be only a few per cent of A_{xz}^N . Without Δ added, R' would be about 20% of A_{xz}^N and C_{xz} would be 20% larger than given by (44').

Assumption (f). The neglect of the cumulant of the (two-time) fourth-order velocity correlation is the most uncertain of the approximations we have made. Such a neglect is basic to most statistical turbulence theories (e.g. Kraichnan 1966; Leslie 1970; Herring 1974). Here, this neglect is partly mitigated in that it is used in a more innocuous way than in the cited turbulence theories. Furthermore, we note that such a neglect is justified by Batchelor (1959, §§8.2 and 8.3) for the energy-containing scales. Another point we wish to stress is that the present neglect of the two-time cumulant is not to be confused with the neglect of a one-time cumulant in quasi-normal theory (e.g. Proudman & Reid 1954). The latter has the defect of causing negative energy spectra, whereas the former does not. Nevertheless, we have not estimated the error in C_{xz} caused by the cumulant neglect – although it may be estimated at a future time.

To conclude our discussion of errors, we note that the cumulant neglect is the main uncertainty of the derivation. The spectral assumptions (a)–(e), together, are estimated to introduce an aggregate uncertainty of about 10% – provided that $E(k) \propto k^{-\frac{5}{3}}$

(for $k > k_L$) at large Reynolds number. However, it is to be understood that the spectral error estimates are based on present limited experimental understanding of $E(k)$ and $E_{xx}(k)$. A more precise understanding of these spectra will allow us to be more confident about the range of possible values of C_{xx} . The main point to be emphasized is that C_{xx} is fairly insensitive to the large-wavelength forms of the spectra $E(k)$ and $S_{xx}(\mathbf{k})$, and, for that reason, can be estimated from a limited knowledge of these forms. A secondary point is that C_{xx} is sensitive to the behaviour of $E(k)$ at large k .

8. Summary

1. (a) Statistical turbulence theory is used to prove that, to lowest order in the anisotropy, an off-diagonal element of the (nonlinear part of the) pressure-strain-rate correlation is proportional to the corresponding element of the stress tensor: $A_{xz}^N + A_{zx}^N = -C_{xz}(\epsilon/\epsilon_0)\langle u_x u_z \rangle$. Higher-order terms in the anisotropy are formally given but not evaluated in this derivation.

(b) It is pointed out that A_{xz}^N is also linearly proportional to other elements of the stress tensor, such as $\langle u_z u_z \rangle$ from the S_{zz} spectrum term in (30), but that the coefficients of these elements are small enough to be neglected. These are contained in R' .

2. An expression for the proportionality constant (Rotta constant) C_{xz} is derived in terms of stress spectra. Hence, the value of C_{xz} can be calculated theoretically from a knowledge of the spectra. This may have practical value because it is difficult to empirically determine C_{xz} in an unambiguous way, or to measure C_{xz} (or A_{xz}^N) directly.

3. (a) The numerical value of C_{xz} is estimated by using theoretical approximations and experimental measurements of stress spectra. This theoretical value of C_{xz} is about 1.6 or 1.7 depending on the long-wavelength behaviour of the spectra.

(b) A principal conclusion of this paper is that C_{xz} is very insensitive to the large-wavelength behaviour of the stress spectra. This insensitivity is attributed to the fact that C_{xz} is actually the ratio of two different time scales [see (36)] both of which depend on the long-wavelength (energy-containing wavelength) behaviour of the spectrum. That behaviour thus tends to cancel out of C_{xz} . As a consequence of this cancellation it has been possible to estimate the uncertainty introduced by those approximations that pertain to the spectra. This estimate is about 10%.

(c) The main uncertainty of the derivation is the neglect of the (two-time) fourth-order velocity cumulant in appendix A.

4. The derivation supports part of Rotta's hypothesis. The verification can only be partial because (38) only applies to one element of the pressure-strain-rate tensor $\rho_0^{-1}\langle p\nabla\mathbf{u}^T \rangle$, whereas a basic aspect of Rotta's hypothesis is that analogous equations hold for all the elements with $C_{xz} = C_{xx} = C_{zz}$. However, the other elements can be readily calculated by the same method.

5. The Rotta constant C_{xz} is quite sensitive to the behaviour of $E(k)$ at large k . This is because C_{xz} is proportional to $k_L v_0^3/\epsilon$, and the latter is easily shown to strongly depend on the large k behaviour of $E(k)$. However, the pressure-strain term A_{xz}^N is not so sensitive to $E(k)$, because it is proportional to ϵC_{xz} and, hence, the ϵ dependence cancels out. In §6, it is assumed that $E(k) \propto k^{-\frac{5}{3}}$ (at large k) when the Reynolds number is large, so that $k_L v_0^3/\epsilon$ is given by (44b). The value of C_{xz} is expected to significantly decrease when the Reynolds number decreases to below about 30 because, then, $k_L v_0^3/\epsilon$ decreases.

Appendix A

In this appendix we calculate the fourth-order velocity correlation $F_{\mathbf{k}}$ and the pressure-strain term A_{zz}^N . This calculation is divided into two parts. In part 1, $F_{\mathbf{k}}$ and A_{zz}^N are derived in terms of two-time velocity covariances $\langle \mathbf{u}_{\mathbf{k}}^*(t_1) \mathbf{u}_{\mathbf{k}}(t) \rangle \equiv \mathbf{S}(\mathbf{k}; t, t_1) V$. In part 2, $\langle \mathbf{u}_{\mathbf{k}}^*(t_1) \mathbf{u}_{\mathbf{k}}(t) \rangle$ is expressed in terms of the single-time covariance $\langle \mathbf{u}_{\mathbf{k}}^*(t) \mathbf{u}_{\mathbf{k}}(t) \rangle \equiv \mathbf{S}(\mathbf{k}) V$, and the t_1 integration of A_{zz}^N is performed.

Part 1

Substituting (8) and (19) into the definition of $F_{\mathbf{k}}$ given by (25), we have

$$ik_x F_{\mathbf{k}} = \int \frac{d\mathbf{k}_1}{(2\pi)^3} \left\{ k_x \mathbf{k} \cdot \langle \mathbf{u}_{\mathbf{k}_1}^*(t_1) \mathbf{u}_{\mathbf{k}_2}^*(t_1) [\mathbf{u}_{\mathbf{k}_3}(t) \mathbf{u}_{\mathbf{k}_4}(t)]' \rangle : \frac{\mathbf{k}^2}{k^2} - k_x k_z \frac{k^2}{k^2} : \langle \mathbf{u}_{\mathbf{k}_1}^*(t_1) \mathbf{u}_{\mathbf{k}_2}^*(t_1) [\mathbf{u}_{\mathbf{k}_3}(t) \mathbf{u}_{\mathbf{k}_4}(t)]' \rangle : \frac{\mathbf{k}^2}{k^2} \right\}, \quad (\text{A } 1)$$

where

$$\mathbf{k}_2 \equiv \mathbf{k} - \mathbf{k}_1, \quad \mathbf{k}_4 \equiv \mathbf{k} - \mathbf{k}_3.$$

To close the fourth-order correlation in (A 1) we first expand it in cumulants as

$$\langle \mathbf{u}_{\mathbf{k}_1}^*(t_1) \mathbf{u}_{\mathbf{k}_2}^*(t_1) [\mathbf{u}_{\mathbf{k}_3}(t) \mathbf{u}_{\mathbf{k}_4}(t)]' \rangle = \langle \mathbf{u}_{\mathbf{k}_1}^*(t_1) \mathbf{u}_{\mathbf{k}_3}(t) \rangle \langle \mathbf{u}_{\mathbf{k}_1}^*(t_1) \mathbf{u}_{\mathbf{k}_4}(t) \rangle + \langle \mathbf{u}_{\mathbf{k}_1}^*(t_1) \mathbf{u}_{\mathbf{k}_4}(t) \rangle \langle \mathbf{u}_{\mathbf{k}_2}^*(t_1) \mathbf{u}_{\mathbf{k}_3}(t) \rangle + Q^{(4)}(t, t_1), \quad (\text{A } 2)$$

where $Q^{(4)}$ is the (fourth-order) cumulant of the correlation on the left-hand side of (A 2). [Note that the term $\langle \mathbf{u}_{\mathbf{k}_1}^*(t_1) \mathbf{u}_{\mathbf{k}_2}^*(t_1) \rangle \langle \mathbf{u}_{\mathbf{k}_3}(t) \mathbf{u}_{\mathbf{k}_4}(t) \rangle$ is not included in the right-hand side because only the fluctuating part of $\mathbf{u}_{\mathbf{k}_3}(t) \mathbf{u}_{\mathbf{k}_4}(t)$ occurs in (A 2). That is (A 2) contains

$$(\mathbf{u}_{\mathbf{k}_3}(t) \mathbf{u}_{\mathbf{k}_4}(t))' \equiv \mathbf{u}_{\mathbf{k}_3}(t) \mathbf{u}_{\mathbf{k}_4}(t) - \langle \mathbf{u}_{\mathbf{k}_3}(t) \mathbf{u}_{\mathbf{k}_4}(t) \rangle.$$

The cumulant $Q^{(4)}$ is a two-time fourth cumulant and is very small for large $(t - t_1)$. Our basic approximation is to neglect $Q^{(4)}$. The neglect of this two-time cumulant is different from, and not as serious as, the neglect of single-time fourth-order cumulants in quasi-normal theory. A similar neglect of two-time cumulants is basic to the direct interaction approximation (Kraichnan 1959; Weinstock 1977, see the discussion in §IV A).

Before substituting (A 2) in (A 1), we express the covariances in (A 2) as

$$\langle \mathbf{u}_{\mathbf{k}_1}^*(t) \mathbf{u}_{\mathbf{k}_3}(t_1) \rangle = \mathbf{S}(k_1; t, t_1) (2\pi)^3 \delta(\mathbf{k}_1 - \mathbf{k}_3), \quad \left. \begin{aligned} \mathbf{S}(\mathbf{k}_1; t, t_1) &\equiv \langle \mathbf{u}_{\mathbf{k}_1}^*(t_1) \mathbf{u}_{\mathbf{k}_1}(t) \rangle V^{-1}, \end{aligned} \right\} \quad (\text{A } 3)$$

where δ is the Dirac delta function, and $\mathbf{S}(\mathbf{k}_1; t, t_1)$ has been normalized with V so as to satisfy

$$\int \frac{d\mathbf{k}_1}{(2\pi)^3} \mathbf{S}(\mathbf{k}_1; t, t_1) = \langle \mathbf{u}(\mathbf{r}, t) \mathbf{u}(\mathbf{r}, t_1) \rangle.$$

Equation (A 3) is valid for homogeneous turbulence, and approximately so for our case of slow variations of average quantities on scales $2\pi k^{-1} \leq L_0$. Substituting (A 3)

and (A 2) in (A 1), neglecting $Q_4(t-t_1)$, and using $\mathbf{k}_2 = \mathbf{k} - \mathbf{k}_1$, $\mathbf{k}_4 = \mathbf{k} - \mathbf{k}_3$, we obtain

$$ik_x F_{\mathbf{k}} = 2V \int \frac{d\mathbf{k}_1}{(2\pi)^3} \left\{ k_x \mathbf{k} \hat{\mathbf{z}} : \mathbf{S}(\mathbf{k}_1; t, t_1) \mathbf{S}(\mathbf{k}_2; t, t_1) : \frac{\mathbf{k}^2}{k^2} - k_x k_z \frac{\mathbf{k}^2}{k^2} : \mathbf{S}(\mathbf{k}_1; t, t_1) \mathbf{S}(\mathbf{k}_2; t, t_1) : \frac{\mathbf{k}^2}{k^2} \right\}, \quad (\text{A } 4)$$

which expresses $F_{\mathbf{k}}$ in terms of two-time velocity covariances $\mathbf{S}(\mathbf{k}; t, t_1)$. The symbol $\hat{\mathbf{z}}$ denotes the unit vector along \mathbf{z} . Substituting (A 4) in the nonlinear part of the pressure-strain term, A_{xz}^* , given by (26), we have

$$A_{xz}^N(t) = \frac{-2}{(1+a)} \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int_0^t dt_1 G_{0\mathbf{k}}^*(t-t_1) \mathbf{b}(\mathbf{k}) : \mathbf{S}(\mathbf{k}_1; t, t_1) \mathbf{S}(\mathbf{k}_2; t, t_1) : \frac{\mathbf{k}^2}{k^2},$$

$$\mathbf{b}(\mathbf{k}) \equiv k_x \mathbf{k} \hat{\mathbf{z}} - \frac{k_x k_z \mathbf{k}^2}{k^2}. \quad (\text{A } 5)$$

Part 2

The purpose of this part of appendix A is to express the two-time covariances in terms of (familiar) single-time covariances. This is not too difficult because, in (A 5), we only need to know $\mathbf{S}(\mathbf{k}; t, t_1)$ for values of $t-t_1$ less than the correlation time τ_E . Here, we define τ_E to be the 'e-folding' time of $\mathbf{S}(\mathbf{k}; t, t_1)$ as follows:

$$\mathbf{S}(\mathbf{k}; t, t-\tau_E) \equiv e^{-1} \mathbf{S}(\mathbf{k}; t, t) \equiv e^{-1} \mathbf{S}(\mathbf{k}). \quad (\text{A } 6)$$

As $t-t_1$ increases beyond τ_E the magnitude of $\mathbf{S}(\mathbf{k}; t, t_1)$ decreases towards zero and, so, does not contribute much to (A 5). Furthermore, much is known about the time dependence of $\mathbf{S}(\mathbf{k}; t, t_1)$ because it is basic to the direct interaction approximation (Kraichnan 1959) and has been widely studied for more than a decade. Several calculations of homogeneous isotropic turbulence (e.g. Kraichnan 1959, 1966; Riley & Patterson 1974; Weinstock 1976) show that $\mathbf{S}(\mathbf{k}; t, t_1)$ is well approximated for $t-t_1 \lesssim \tau_E$, and $\mathbf{U} = 0$, by

$$\mathbf{S}(\mathbf{k}; t, t_1) = \mathbf{S}(\mathbf{k}) \exp \left[-\frac{1}{2} k^2 v_0^2 (t-t_1)^2 \right], \quad t-t_1 \leq \tau_E, \quad (\text{A } 7)$$

$$\mathbf{S}(\mathbf{k}) = \mathbf{S}(\mathbf{k}; t, t), \quad (\text{A } 8)$$

where $v_0^2 = \frac{1}{3} \langle \mathbf{u} \cdot \mathbf{u} \rangle$. It is seen that $\mathbf{S}(\mathbf{k})$ is just the ordinary (single-time) spectrum of the velocity field at time t . It is related to the Reynolds stress by

$$\langle \mathbf{u} \mathbf{u} \rangle = \int \frac{d\mathbf{k}}{(2\pi)^3} \mathbf{S}(\mathbf{k}). \quad (\text{A } 9)$$

It can be shown that for anisotropic turbulence (A 7) should be

$$\mathbf{S}(\mathbf{k}; t, t_1) = \mathbf{S}(\mathbf{k}) \exp \left[-\frac{1}{2} \mathbf{k} \mathbf{k} : \langle \mathbf{u} \mathbf{u} \rangle (t-t_1)^2 \right]$$

(see equation (53) of Weinstock 1976). For the present case where $\mathbf{U} \neq 0$, this equation is easily generalized to

$$\mathbf{S}(\mathbf{k}; t, t_1) = \mathbf{S}(\mathbf{k}) \exp \left[-\frac{1}{2} \mathbf{k} \mathbf{k} : \langle \mathbf{u} \mathbf{u} \rangle (t-t_1)^2 - i \mathbf{k} \cdot \mathbf{U} (t-t_1) \right] \quad (t-t_1 \leq \tau_E), \quad (\text{A } 10)$$

assuming small variation of \mathbf{U} on the scale L_0 . The magnitude of $\mathbf{S}(\mathbf{k}; t, t_1)$ monotonically decreases with increasing $t-t_1$, although the rate of decrease is slower than

that of (A 10) when $t - t_1 > \tau_E$ (e.g. Kraichnan 1959, 1966). However, when $t - t_1 > \tau_E$, the magnitude of $\mathbf{S}(\mathbf{k}; t, t_1)$ is so small that (A 10) can be used in (A 5) for all $t - t_1$ with little error.

We wish to use (A 10), in (A 5), even though it has only been tested for homogeneous turbulence. This use can be justified as consistent with our assumption of weak inhomogeneity. We therefore substitute (A 10) into (A 5). We also substitute (20*b*) into (A 5) to obtain

$$A_{xz}^N = -2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \mathbf{b}(\mathbf{k}) : \mathbf{S}(\mathbf{k}_1) \mathbf{S}(\mathbf{k}_2) : \frac{\mathbf{k}^2}{k^2} (1+a)^{-1} \times \int_0^t dt_1 \exp[-\frac{1}{2}(\mathbf{k}_1^2 + \mathbf{k}_2^2) : \langle \mathbf{u}\mathbf{u} \rangle (t-t_1)^2],$$

which becomes for large t , $t \gg [(\mathbf{k}_1^2 + \mathbf{k}_2^2) : \langle \mathbf{u}\mathbf{u} \rangle]^{-1}$,

$$A_{xz}^N = -2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \frac{\tau_c}{1+a} \mathbf{b}(\mathbf{k}) : \mathbf{S}(\mathbf{k}_1) \mathbf{S}(\mathbf{k}_2) : \frac{\mathbf{k}^2}{k^2}, \tag{A 11}$$

$$\tau_c \equiv (\pi/2)^{\frac{1}{2}} [(\mathbf{k}_1^2 + \mathbf{k}_2^2) : \langle \mathbf{u}\mathbf{u} \rangle]^{-\frac{1}{2}}.$$

Note that the \mathbf{U} term in $G_{0\mathbf{k}}^*(t-t_1)$ has cancelled out the $i\mathbf{k} \cdot \mathbf{U}$ term in (A 10) so that $A_{xz}^N(t)$ does not depend on \mathbf{U} in an explicit way. Equation (A 11) is the desired expression for A_{xz}^N in terms of single-time velocity covariances \mathbf{S} .

Appendix B

To perform the spherical integrations in (37), it is convenient to divide it into four parts by dividing γ into four parts. To obtain the desired division we use $k_x k_{2x} = k_{2x}^2 + k_{1x} k_{2x}$ and $k_{1z} + k_{2z} \equiv k_z$. We are thus able to write (33) for γ as

$$\left. \begin{aligned} \gamma &\equiv \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \\ \gamma_1 &\equiv \frac{k_{2x}^2 [k_1^2 - (\mathbf{k}_1 \cdot \hat{\mathbf{k}}_2)^2]}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2}, \\ \gamma_2 &\equiv \frac{(k_{1x} k_{2x} - k_x k_{1x} k_{2y} k_{1y}^{-1}) [k_1^2 - (\mathbf{k}_1 \cdot \hat{\mathbf{k}}_2)^2]}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2}, \\ \gamma_3 &\equiv -\frac{4k_x k_z k_1^2 (1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{k^2} \left[\frac{k_{2x} k_{2z} - k_{2y} k_{1y}^{-1} (k_{2x} k_{1z} + k_{1x} k_{2z})}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2} \right], \\ \gamma_4 &\equiv 2 \left\{ k_x k_z (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) \left[\frac{4k_1^2 (1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)}{k^2} - \frac{k_1}{k_2} \right] \right. \\ &\quad + k_x k_{1z} \left(1 + \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_2^2} \right) \left[\frac{k_{2x} k_{2z} - k_{2y} k_{1y}^{-1} (k_{2x} k_{1z} + k_{1x} k_{2z})}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2} \right] \\ &\quad \left. - k_{1x} k_{1z} \left\{ \frac{2[k_1^2 - (\mathbf{k}_1 \cdot \hat{\mathbf{k}}_2)^2]}{k^2} - 1 \right\} \frac{2k_{2y} k_{1y}^{-1} (k_{2x} k_{1z} + k_{1x} k_{2z})}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2} [1 - \frac{1}{2}(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2] \right\}, \end{aligned} \right\} \tag{B 1}$$

where to separate γ_3 from the other parts of γ we used

$$[k_1^2 - (\mathbf{k}_1 \cdot \hat{\mathbf{k}}_2)^2] \equiv k_1^2 [1 - (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2] = k_1^2 [(1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2 - 2(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) (1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)].$$

The purpose of the division of γ in (B 1) is to cause an approximate cancellation of $\mathbf{k}_1 \mathbf{k}_2$ from the factors $k_1^2 - (\mathbf{k}_1 \mathbf{k}_2)^2$ and $\mathbf{k}^2 \equiv k_1^2 + k_2^2 + 2\mathbf{k}_1 \mathbf{k}_2$ in γ_1 , γ_2 , and γ_3 . These cancellations are described by (B 5) and (B 13).

Substitution of (B 1) in (37) thus yields the four parts of C_{zz} :

$$C_{zz} \equiv C^{(1)} + C^{(2)} + C^{(3)} + C^{(4)}, \tag{B 2}$$

$$C^{(i)} \equiv 6 \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{\gamma_i E(k_2) S_{zz}(\mathbf{k}_1)}{k_2^2 e_0 \langle u_x u_z \rangle}, \quad i = 1, 2, 3, 4.$$

We calculate $C^{(1)}$ first because it is simplest, and, we find, largest. It is given by (B 2) and (B 1) as

$$C^{(1)} \equiv 6 \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{k_{2x}^2 [k_1^2 - (\mathbf{k}_1 \cdot \mathbf{k}_2)^2] E(k_2) S_{zz}(\mathbf{k}_1)}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2 k_2^2 e_0 \langle u_x u_z \rangle}. \tag{B 3}$$

Let θ denote the angle that \mathbf{k}_1 makes with \mathbf{k}_2 :

$$\mathbf{k}_2 \cdot \mathbf{k}_1 = k_1 k_2 \cos \theta.$$

The main dependence of the integrand of (B 3) on θ is given, with $k^2 \equiv |\mathbf{k}_1 + \mathbf{k}_2|^2$, by

$$\frac{[k_1^2 - (\mathbf{k}_1 \cdot \mathbf{k}_2)^2]}{k_1^2 + k_2^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2} = \frac{k_1^2 (1 - \cos^2 \theta)}{k_1^2 + k_2^2 + 2k_1 k_2 \cos \theta}. \tag{B 4}$$

The chief assumption we shall make to evaluate $C^{(1)}$, as well as the other $C^{(i)}$, is that the main contribution to the (scalar) k_1 and k_2 integrations in (B 3) come from $k_1 \approx k_2$. This assumption leads to an error of about 2% (see §7d) and greatly simplifies the integrations in (B 2). The basis of this assumption is that the main contribution comes from the energy-containing region in the vicinity of which $E(k_2)$ and $S_{zz}(\mathbf{k}_1)$ have their maximum values. It is then assumed that $S_{zz}(\mathbf{k}_2)$ attains its maximum at approximately the same wavenumber as does $E(k_2)$. This assumption is tested and justified in appendix C with the experimental data of Champagne *et al.* (1970) and Kaimal *et al.* (1972). Hence (B 4) can be approximated by

$$\begin{aligned} \frac{[k_1^2 - (\mathbf{k}_1 \cdot \mathbf{k}_2)^2]}{k_1^2 + k_2^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2} &\approx \frac{k_1^2 (1 - \cos^2 \theta)}{(k_1^2 + k_2^2) (1 + \cos \theta)} \\ &= \frac{k_1^2 (1 - \cos \theta)}{(k_1^2 + k_2^2)}. \end{aligned} \tag{B 5}$$

When (B 5) is substituted into the integrand of (B 3), the $\cos \theta$ vanishes compared with unity because the remainder of the integrand is independent of θ ; that is, we have $\int d\mathbf{k}_1 \int d\mathbf{k}_2 \cos \theta \approx 0$ when the $\cos \theta$ part of (B 5) is substituted in (B 3). Substitution of (B 5) in (B 3) thereby yields

$$C^{(1)} = 6 \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{e_0 v_0}{\epsilon}\right) \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{k_{2x}^2 k_1^2 E(k_2) S_{zz}(\mathbf{k}_1)}{(k_1^2 + k_2^2)^{\frac{3}{2}} k_2^2 e_0 \langle u_x u_z \rangle}. \tag{B 6}$$

We next express the \mathbf{k}_1 and \mathbf{k}_2 integrals in spherical co-ordinates, e.g.

$$\int d\mathbf{k}_2 \equiv \int_0^\infty k_2^2 dk_2 \int_0^\pi d\theta_2 \sin \theta_2 \int_0^{2\pi} d\phi_2,$$

where θ_2 is the angle \mathbf{k}_2 makes with the $\hat{\mathbf{x}}$ axis ($k_{2x} = k_2 \cos \theta_2$), and ϕ_2 is the (azimuthal) angle of \mathbf{k}_2 in the plane perpendicular to \mathbf{x} . We then perform the θ_2 and ϕ_2 integrations in (B 6) as follows:

$$\frac{1}{4\pi} \int_0^{2\pi} d\phi_2 \int_0^\pi d\theta_2 \sin \theta_2 k_{2x}^2 = \frac{1}{3} k_2^2. \quad (\text{B } 7)$$

We also integrate $S_{xz}(\mathbf{k}_1)$ over a spherical shell of radius k_1 to obtain a scalar spectrum, which we denote by $E_{xz}(\mathbf{k}_1)$:

$$E_{xz}(k_1) \equiv \frac{k_1^2}{(2\pi)^3} \int_0^{2\pi} d\phi_1 \int_0^\pi d\theta_1 \sin \theta_1 S_{xz}(\mathbf{k}_1), \quad (\text{B } 8)$$

where θ_1 is the angle \mathbf{k}_1 makes with the \mathbf{x} axis and ϕ_1 is the azimuthal angle of \mathbf{k}_1 . Substitution of (B 7) and (B 8) in (B 6) finally gives the scalar integral:

$$C^{(1)} = 2 \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_2^2 E(k_2) E_{xz}(k_1) k_1^2}{(k_1^2 + k_2^2)^{\frac{3}{2}} e_0 \langle u_x u_z \rangle}. \quad (\text{B } 9)$$

Next, we evaluate $C^{(2)}$. The expression for $C^{(2)}$ is given by substitution of γ_2 in (B 2):

$$C^{(2)} = 6 \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{k_{1x}(k_{2x} - k_x k_{2y} k_{1y}^{-1}) [k_1^2 - (\mathbf{k}_1 \cdot \mathbf{k}_2)^2] E(k_2) S_{xz}(\mathbf{k}_1)}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2 k_2^2 e_0 \langle u_x u_z \rangle}. \quad (\text{B } 10)$$

This integrates very easily if we use approximation (B 5) as justified in appendix C and §7(b). The factor in (B 10) then becomes

$$\begin{aligned} k_{1x}(k_{2x} - k_x k_{2y} k_{1y}^{-1}) \frac{[k_1^2 - (\mathbf{k}_1 \cdot \mathbf{k}_2)^2]}{k^2} &\approx k_{1x} \left(k_{2x} - \frac{k_x k_{2y}}{k_{1y}} \right) \frac{k_1^2 (1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)}{k_1^2 + k_2^2} \\ &= k_{1x} \left[k_{2x} - \frac{(k_{1x} + k_{2x}) k_{2y}}{k_{1y}} \right] \left[1 - \frac{(k_{1x} k_{2x} + k_{1y} k_{2y} + k_{1z} k_{2z})}{k_1 k_2} \right] \frac{k_1^2}{k_1^2 + k_2^2}, \end{aligned} \quad (\text{B } 11)$$

where we have used the cancellation $[1 - (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2]/(1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) = 1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2$. We substitute (B 11) in (B 10) and note that all terms odd in k_{2x} , k_{2y} or k_{2z} vanish because $E(k_2)$ is an even function of these components. Hence, (B 12) reduces to

$$C^{(2)} = 6 \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{k_{1x}^2 k_{1y}^2 (k_{2y}^2 - k_{2x}^2) E(k_2) S_{xz}(\mathbf{k}_1)}{(k_1^2 + k_2^2)^{\frac{1}{2}} k^2 k_2^2 e_0 \langle u_x u_z \rangle} = 0. \quad (\text{B } 12)$$

The right-hand side of (B 12) vanishes because $E(k_2)$ is a scalar isotropic function of k_2 so that the integrals over k_{2x}^2 and k_{2y}^2 cancel each other out.

The expression for $C^{(3)}$ is given by substituting γ_3 , from (B 1), into (B 2). The expression for γ_3 is greatly simplified by using

$$\frac{(1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{(k_1^2 + k_2^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2)^2} \approx \frac{(1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{(k_1^2 + k_2^2)^2 (1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2} = \frac{1}{(k_1^2 + k_2^2)^2}, \quad (\text{B } 13)$$

where we have again used the approximation $k_1 \approx k_2$, justified in appendix C. Substitution of γ_3 and (B 13) in (B 2) yields $C^{(3)}$ as

$$\begin{aligned} C^{(3)} &\approx -6 \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{[k_{2x} k_{2z} - k_{2y} k_{1y}^{-1} (k_{2x} k_{1x} + k_{1x} k_{2x})]}{(k_1^2 + k_2^2)^{\frac{3}{2}}} \\ &\quad \times \frac{4k_1^2 k_x k_z E(k_2) S_{xz}(\mathbf{k}_1)}{k_2^2 e_0 \langle u_x u_z \rangle}. \end{aligned} \quad (\text{B } 14)$$

Those terms in (B 14) which are odd in k_{2x} , k_{2y} or k_{2z} vanish, so that (using $k_x = k_{1x} + k_{2x}$, $k_z = k_{1z} + k_{2z}$) (B 14) reduces to

$$C^{(3)} = 6 \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int \frac{d\mathbf{k}_1}{(2\pi)^3} \int \frac{d\mathbf{k}_2}{4\pi} \frac{4k_1^2 k_{2x}^2 k_{2z}^2 E(k_2) S_{xz}(\mathbf{k}_1)}{(k_1^2 + k_2^2)^{\frac{3}{2}} k_2^2 e_0 \langle u_x u_z \rangle}. \quad (\text{B } 15)$$

The \mathbf{k}_1 and \mathbf{k}_2 integrals are next expressed in spherical co-ordinates as was done for $C^{(1)}$. The θ_2 and ϕ_2 integrations in (B 15) are given by

$$\frac{1}{4\pi} \int_0^{2\pi} d\phi_2 \int d\theta_2 \sin \theta_2 k_{2x}^2 k_{2z}^2 = \frac{1}{15} k_2^4, \quad (\text{B } 16)$$

and the ϕ_1 and θ_1 integrations are the same as (B 8). Substitution of (B 8) and (B 16) in (B 15) gives $C^{(3)}$ as

$$C^{(3)} = -\frac{8}{5} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_1^2 k_2^4 E(k_2) E_{xz}(k_1)}{(k_1^2 + k_2^2)^{\frac{3}{2}} e_0 \langle u_x u_z \rangle}. \quad (\text{B } 17)$$

This expression is similar in form to that given by (B 9) for $C^{(1)}$. The integrand of (B 17) contains the additional factor $k_2^2(k_1^2 + k_2^2)^{-1}$. However, as pointed out for (B 3), the main contribution to the integrations in (B 17) come from $k_1 \approx k_2$, assuming that $E_{xz}(k)$ has its maximum value near the value of k for which $E(k)$ has its maximum. (This assumption is tested and justified in §7(b) and appendix C.) In that case we could take $2k_1^2(k_1^2 + k_2^2)^{-1} \approx 1$ in (B 17) and, using (B 9), we obtain

$$C^{(3)} \approx -\frac{2}{5} C^{(1)}. \quad (\text{B } 18)$$

We finally come to the evaluation of $C^{(4)}$, given by substitution of γ_4 in (B 2). It is seen that $C^{(4)}$ is more complex than $C^{(1)}$, $C^{(2)}$ or $C^{(3)}$ because it contains several terms in the integrand. However, there is a great simplification when approximation (B 5) is used to reduce γ_4 . Furthermore, $C^{(4)}$ is also much smaller than $C^{(1)}$. We will not present the calculation of $C^{(4)}$ here because it is lengthy and $C^{(4)}$ is small. We will only quote the result as follows:

$$C^{(4)} \approx -\frac{1}{9} C^{(1)}. \quad (\text{B } 19)$$

Among the approximations used to obtain (B 19) are the substitution of (B 5) in factors of γ_4 as

$$\begin{aligned} \frac{2[k_1^2 - (\mathbf{k}_1 \cdot \mathbf{k}_2)^2]}{k_1^2 + k_2^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2} - 1 &\approx \frac{2k_1^2(1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)}{k_1^2 + k_2^2} - 1, \\ \frac{4k_1^2(1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)}{k_1^2 + k_2^2 + 2\mathbf{k}_1 \cdot \mathbf{k}_2} - \frac{k_1}{k_2} &\approx \frac{4k_1^2}{k_1^2 + k_2^2} - \frac{k_1}{k_2}, \end{aligned}$$

and the expansion of k^{-2} as

$$k^{-2} \approx (k_1^2 + k_2^2)^{-1} (1 + \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^{-1} \approx \frac{1 - (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2) + (\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)^2}{k_1^2 + k_2^2}.$$

The terms higher than second order in $(\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2)$ are found to give an extremely small contribution to $C^{(4)}$.

Finally, C_{xz} is obtained by substituting (B 9), (B 12), (B 18), and (B 19) in (B 2)

$$C_{xz} = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int_0^\infty dk_1 \int_0^\infty dk_2 \frac{k_2^2 k_1^2 E(k_2) E_{xz}(k_1)}{(k_1^2 + k_2^2)^{\frac{3}{2}}} e_0 \langle u_x u_z \rangle. \quad (\text{B } 20)$$

As to the accuracy of (B 20), we recall that it is based on the approximations (B 5) and use of $2k_1^2(k_1^2 + k_2^2)^{-1} \approx 1$. These approximations are discussed in §7(b).

Appendix C

C1. Theoretical variation of C_{xz} with k'_L/k_L

To determine what happens to C_{xz} when the maximum of $E_{xz}(k)$ occurs at a wave-number different from that of the maximum of $E(k)$, we will use a very simple delta function model of E and E_{xz} :

$$E(k) = e_0 \delta(k - k_L), \quad E_{xz}(k) = \langle u_x u_z \rangle \delta(k - k'_L), \quad (\text{C } 1)$$

which satisfies (28') and (40'). This model isolates, and, perhaps, exaggerates how rapidly C_{xz} varies with k'_L/k_L . To determine this variation we substitute (B 9), (B 12), (B 17), and (B 19) into (B 2) to obtain C_{xz} in the form

$$C_{xz} \equiv C^{(1)} + C^{(2)} + C^{(3)} + C^{(4)} \\ = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0}{\epsilon}\right) \int_0^\infty dk_1 \int_0^\infty dk_2 \frac{E(k_2) E_{xz}(k_1) k_1^2 k_2^2}{(k_1^2 + k_2^2)^{\frac{3}{2}}} e_0 \langle u_x u_z \rangle \left[\frac{16}{9} - \frac{8k_1^2/5}{k_1^2 + k_2^2} \right]. \quad (\text{C } 2)$$

This expression is more rigorous than (39) for cases in which $k_L/k'_L \neq 1$ because (B 17) is used instead of the more approximate (B 18).

Equation (C 2) reduces to (39) when the square-bracketed term is approximated with $k_1 \approx k_2$. The variation of C_{xz} with k'_L/k_L can now be estimated by substituting (C 1) in (C 3) to obtain

$$C_{xz} = \frac{16}{9} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0 k_L}{\epsilon}\right) \left[\frac{y}{(1+y)^{\frac{3}{2}}} \left(1 - \frac{0.9y}{1+y}\right) \right], \quad (\text{C } 3)$$

$$y \equiv (k'_L/k_L)^2. \quad (\text{C } 4)$$

For $k'_L/k_L = 1$, C_{xz} is given by

$$C_{xz} = \frac{16}{9} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{v_0 e_0 k_L}{\epsilon}\right) (0.55) \times 2^{-\frac{3}{2}} \quad (k'_L = k_L). \quad (\text{C } 5)$$

When $(k'_L/k_L) = 0.5, 0.7, 0.8,$ and 2.0 , then it is found that C_{xz} is less than (C 5) by 25, 3, 0, and 25%, respectively.

C2. Experimental estimate of k'_L/k_L

Two experiments we refer to are by Champagne *et al.* (1970) and that analysed by Kaimal *et al.* (1972). However, both experiments provide measurements of the one-dimensional spectra $S_{ii}(k_x)$ and $S_{xz}(k_x)$ and not of the three-dimensional scalar spectra $E(k)$ and $E_{xz}(k)$ needed by us. We must therefore rely on crude approxima-

tions. For example, the spectrum $E(k)$ will be approximated by using the isotropic relation (e.g. Tennekes & Lumley 1972)

$$E(k) \approx k^3 \frac{d}{dk} \left[\frac{1}{k} \frac{dS_{xx}(k)}{dk} \right]. \tag{C 6}$$

Champagne *et al.* (1970) used an isotropic relation for S_{zz} in terms of S_{xx} and found errors (factors of 2) only at small k . The isotropic relation (C 6) is not expected to be too bad because the shapes of the experimental $S_{xx}(k_x)$, $S_{zz}(k_x)$, and $S_{yy}(k_x)$ spectra are very similar to each other. Furthermore, the errors introduced by (C 6) will be partly cancelled out by the fact that we only use the ratio of k'_L to k_L , not their individual values. The ratio of E_{zz} to E can be written as

$$\frac{E_{zz}(k)}{E(k)} = D_1 k^{-n(k)}, \tag{C 7}$$

where D_1 is a (normalization) constant, and the exponent $n(k)$ varies with k . Only the value of $n(k)$ in the vicinity of $k = k_L$, the peak of $E(k)$, is needed. To estimate $n(k_L)$ we use figure 22 of Champagne *et al.*, which gives the spectral ratio $S_{zz}(k_x) [S_{xx}(k_x) S_{zz}(k_x)]^{-\frac{1}{2}}$ as a function of k_x . This ratio can be represented by

$$\frac{S_{zz}(k)}{[S_{xx}(k) S_{zz}(k)]^{\frac{1}{2}}} = D_2 k^{-m(k)}, \tag{C 8}$$

where D_2 , a constant, and $m(k)$ are determined by the data. To estimate $n(k)$ we assume

$$n(k) \approx m(k). \tag{C 9}$$

From figure 22 of Champagne *et al.*, it is seen that $m(k) \approx 0.3$ for k in the vicinity of k_L where $E(k)$ is maximum; this maximum is estimated to occur at the abscissa $nk_1 = 10^{-2}$ (by using $dE(k)/dk = 0$ and (C 6) with figure 19 of Champagne *et al.*). Thus, with (C 9), we approximate $n(k) \approx 0.3$ for $k \approx k_L$. The maximum of $E_{zz}(k)$ is determined from (C 7) by setting $dE_{zz}(k) = 0$ and using $n(k) \approx n(k_L) \approx 0.3$. The resulting value of k'_L is given by

$$k'_L \approx 0.7k_L \tag{C 10}$$

(Champagne *et al.*) The most serious source of error in this calculation is that the experimental data is differentiated three times in $dE(k)/dk$.

The experimental data of Kaimal *et al.* (1972) can also be used to estimate k'_L/k_L . This estimate is made from their empirical formulas of S_{xz} , S_{xx} , S_{zz} for neutral lapse rates given in their §7. From the formula for $S_{xx}(k_x)$ we use (C 6) to estimate that the maximum of $E(k)$ occurs at $f = 0.028$ (f is the frequency used in the formulas), and from the formulas for S_{xz} , S_{xx} , and S_{zz} we estimate $n(k_L) = 0.25$. With this value of $n(k_L)$ we use (C 7) and $dE_{zz}/dk = 0$ to determine k'_L as

$$k'_L \approx 0.7k_L \tag{C 11}$$

in agreement with the estimate for the data of Champagne *et al.* We should note that the data of Kaimal *et al.* are more anisotropic than that of Champagne *et al.* because

the maxima of $S_{zz}(k_x)$ and $S_{xx}(k_x)$ occur at markedly different wavenumbers. For that reason, our application of (C 6) to the data of Kaimal *et al.* is more questionable.

Appendix D. Low Reynolds number

Equation (27) for the nonlinear part, A_{zz}^N , of the pressure-strain rate was derived for large Reynolds number. This restriction is caused by the neglect of the molecular viscosity term $\nu \nabla^2 u_z$ in (17). It is not difficult to correct (27) so as to apply to arbitrary Reynolds number. To do so, we first rewrite (12), or (2), with the viscosity term on the left side:

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla - \nu \nabla^2\right) \mathbf{u}(t) = \mathbf{I}^0(t), \tag{D 1}$$

$$\mathbf{I}^0(t) \equiv -[\mathbf{u}(t) \cdot \nabla \mathbf{u}(t)]' - \mathbf{u} \cdot \nabla \mathbf{U} - \frac{\nabla p(t)}{\rho_0}.$$

The formal solution of (D 1) is

$$\mathbf{u}(t) = \int d\mathbf{r}_1 G^v(\mathbf{r}, t; \mathbf{r}_1, 0) \mathbf{u}(\mathbf{r}_1, 0) + \int_0^t dt_1 \int d\mathbf{r}_1 G^v(\mathbf{r}, t; \mathbf{r}_1, t_1) \mathbf{I}^0(\mathbf{r}_1, t_1), \tag{D 2}$$

where G^v is an operator defined, in analogy with (14), by

$$\left. \begin{aligned} \left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla - \nu \nabla^2\right) G^v(\mathbf{r}, t; \mathbf{r}_1, t_1) &= 0, \\ G^v(\mathbf{r}, t_1; \mathbf{r}_1, t_1) &= \delta(\mathbf{r} - \mathbf{r}_1). \end{aligned} \right\} \tag{D 3}$$

The equations following (16) can now be corrected to include the $\nu \nabla^2 \mathbf{u}$ dissipation term by replacing G_0 with G^v . Actually, what is needed is a Fourier component $G_{\mathbf{k}}^v$ (defined by)

$$G_{\mathbf{k}}^v \equiv \int d(\mathbf{r} - \mathbf{r}_1) G^v(\mathbf{r}, t; \mathbf{r}_1, t_1) \exp[-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_1)]. \tag{D 4}$$

If \mathbf{U} is slowly varying in time, then

$$G^v = \exp[-(t - t_1)(\mathbf{U} \cdot \nabla - \nu \nabla^2)] \delta(\mathbf{r} - \mathbf{r}_1), \tag{D 5}$$

and, if \mathbf{U} varies slowly in space on the scale k^{-1} , then (D 4) becomes

$$G_{\mathbf{k}}^v \approx \exp[-(t - t_1)(i\mathbf{k} \cdot \mathbf{U} + \nu k^2)]. \tag{D 6}$$

It can be seen that the expressions following (16) can be corrected by everywhere replacing $G_{0\mathbf{k}}$ with $G_{\mathbf{k}}^v$. In addition, (A 10) for $\mathbf{S}(\mathbf{k}; t, t_1)$ must also be corrected for the viscosity term. This correction can be shown to be obtained by adding $-\nu k^2(t - t_1)$ to (A 10) as follows:

$$\mathbf{S}(\mathbf{k}; t, t_1) = \mathbf{S}(\mathbf{k}) \exp\left[-\frac{1}{2}\mathbf{k}^2 : \langle \mathbf{u}\mathbf{u} \rangle (t - t_1)^2 - i\mathbf{k} \cdot \mathbf{U}_0(t - t_1) - \nu k^2(t - t_1)\right]. \tag{D 7}$$

Returning our attention to (27), we find that A_{zz}^N contains $G_{0\mathbf{k}}$ and $\mathbf{S}(\mathbf{k}; t, t_1)/\mathbf{S}(\mathbf{k})$ only in the characteristic time τ_c , which was given [see (A 11)] by

$$\tau_c \equiv \int_0^t dt_1 \exp\left[-\frac{1}{2}(\mathbf{k}_1^2 + \mathbf{k}_2^2) : \langle \mathbf{u}\mathbf{u} \rangle (t - t_1)^2\right], \quad t \sim \infty. \tag{D 8}$$

When $G_{0\mathbf{k}}$ is replaced by $G_{\mathbf{k}}^v$, and $\mathbf{S}(\mathbf{k}; t, t_1)$ is replaced by (D 7), then it can be shown that τ_c in (A 11) becomes τ_c^v , given by

$$\tau_c^v \equiv \int_0^t dt_1 \exp\left[-\frac{1}{2}(\mathbf{k}_1^2 + \mathbf{k}_2^2) : \langle \mathbf{u}\mathbf{u} \rangle (t - t_1)^2 - 2\nu k^2(t - t_1)\right]. \tag{D 9}$$

Hence, (27) for A_{zz}^N is generalized to apply to all Reynolds number by replacing τ_c with τ_c^v :

$$A_{zz}^N = -2 \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\mathbf{k}_1}{(2\pi)^3} \frac{\tau_c^v \mathbf{b}(\mathbf{k})}{(1+a)} : \mathbf{S}(\mathbf{k}_2) \mathbf{S}(\mathbf{k}_1) : \frac{\mathbf{k}^2}{k^2}. \quad (\text{D } 10)$$

We thus see that the effect of including molecular viscosity is explicitly accounted for by adding $-2\nu k^2$ in the expression for the characteristic time τ_c . In general, we believe that the Lagrangian integral time scales must likewise be corrected with $-2\nu k^2$ when the Reynolds number is low.

REFERENCES

- BACHELOR, G. K. 1959 *Homogeneous Turbulence*. Cambridge University Press.
- CHAMPAGNE, F. H., HARRIS, V. G. & CORRSIN, S. 1970 Experiments on nearly homogeneous turbulent shear flow. *J. Fluid Mech.* **41**, 81.
- CHOU, P. Y. 1945 On velocity correlations and the solutions of the equations of turbulent fluctuation. *Quart. Appl. Math.* **3**, 38.
- COMTE-BELLOT, G. & CORRSIN, S. 1966 The use of a contraction to improve the isotropy of grid generated turbulence. *J. Fluid Mech.* **25**, 657-682.
- EDWARDS, S. F. 1964 The theoretical dynamics of homogeneous turbulence. *J. Fluid Mech.* **18**, 239.
- HANJALIC, K. & LAUNDER, B. E. 1972 A Reynolds stress model of turbulence and its application to thin shear flows. *J. Fluid Mech.* **52**, 609-638.
- HARRIS, V. G., GRAHAM, J. A. H. & CORRSIN, S. 1977 Further experiments in nearly homogeneous turbulent shear flow. *J. Fluid Mech.* **81**, 657-687.
- HERRING, J. R. 1965 Self-consistent field approach to turbulence theory. *Phys. Fluids* **8**, 2219-2225.
- HERRING, J. R. 1973 Statistical turbulence theory and turbulence phenomenology. *Proc. NASA-Langley Conf. for Turbulent Shear Flow*.
- HERRING, J. R. 1974 Approach of axisymmetric turbulence to isotropy. *Phys. Fluids* **17**, 859-872.
- KAIMAL, J. C., WYNGAARD, J. C., IZUMI, Y. & COTE, O. R. 1972 Spectral characteristics of surface-layer turbulence. *Quart. J. Roy. Met. Soc.* **98**, 563-589.
- KRAICHNAN, R. 1959 The structure of isotropic turbulence at high Reynolds numbers. *J. Fluid Mech.* **5**, 497-543.
- KRAICHNAN, R. 1966 Isotropic turbulence and inertial-range structure. *Phys. Fluids* **9**, 1728-1752.
- LAUNDER, B. E., REECE, G. J. & RODI, W. 1975 Progress in the development of a Reynolds-stress turbulence closure. *J. Fluid Mech.* **68**, 537-566.
- LESLIE, D. C. 1970 Simplification of the direct interaction equations for turbulent shear flow. *J. Phys.* A **3**, L16-L18.
- LESLIE, D. C. 1973 *Developments in the Theory of Turbulence*. Clarendon.
- LUMLEY, J. L. & KHAJEK-NOURI, B. 1974 Computational modeling of turbulent transport. *Adv. Geophys.* A **18**, 169-192.
- LUMLEY, J. L. & NEWMAN, G. R. 1977 The return to isotropy of homogeneous turbulence. *J. Fluid Mech.* **82**, 161-178.
- ORSZAG, S. A. 1970 Analytical theories of turbulence. *J. Fluid Mech.* **41**, 363.
- PANOFSKY, H. A. & MARES, E. 1968 Recent measurements of cospectra for heat flux and stress. *Quart. J. Roy. Met. Soc.* **94**, 581-584.
- PROUDMAN, I. & REID, W. H. 1954 On the decay of a normally distributed and homogeneous turbulent field. *Phil. Trans. Roy. Soc. A* **247**, 163.
- REYNOLDS, W. C. 1976 Computation of turbulent flows. *Ann. Rev. Phys.* **8**, 183-208.
- RILEY, J. J. & PATTERSON, G. S. 1974 Diffusion experiments with numerically integrated isotropic turbulence. *Phys. Fluids* **17**, 292-297.

- ROTTA, J. 1951 Statistische Theorie nichthomogener Turbulenz. 1. Mitteilung. *Z. Phys.* **129**, 547–572.
- SCHUMANN, V. & HERRING, J. R. 1976 Axisymmetric homogeneous turbulence: a comparison of direct spectral simulations with the direct-interaction approximation.
- TENNEKES, H. & LUMLEY, J. L. 1972 *A First Course in Turbulence*. M.I.T. Press.
- WEINSTOCK, J. 1976 Lagrangian–Eulerian relation and the independence approximation. *Phys. Fluids* **19**, 1702–1711.
- WEINSTOCK, J. 1977 Three-point method in the theory of homogeneous turbulence. *Phys. Fluids* **20**, 1631–1650.